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Performance evaluation for communication systems with receive diversity and interference

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ABSTRACT

PERFORMANCE EVALUATION FOR COMMUNICATION SYSTEMS WITH RECEIVE DIVERSITY AND INTERFERENCE

by
Debang Lao

Optimum combining (OC) is a well-known coherent detection technique used to combat fading and suppress cochannel interference. In this dissertation, expressions are developed to evaluate the error probability of OC for systems with multiple interferers and multiple receiving branches. Three approaches are taken to derive the expressions. The first one starts from the decision metrics of OC. It facilitates obtaining closed-form expressions for binary phase-shift keying modulation. The second approach utilizes the moment generating function of the output signal to interference plus noise ratio (SINR) and results in expressions for symbol and bit error probability for multiple phase-shift keying modulation. The third method uses the probability density function of the output SINR and arrives at expressions of symbol error probability for systems where the interferers may have unequal power levels. Throughout the derivation, it is assumed that the channels are independent Rayleigh fading channels. With these expressions, evaluating the error probability of OC is fast, easy and accurate.

Two noncoherent detection schemes based on the multiple symbol differential detection (MSDD) technique are also developed for systems with multiple interferers and multiple receiving branches. The first MSDD scheme is developed for systems where the channel gain of the desired signal is unknown to the receiver, but the covariance matrix of the interference plus noise is known. The maximum-likelihood decision statistic is derived for the detector. The performance of MSDD is demonstrated by analysis and simulation. A sub-optimum decision feedback algorithm is presented to reduce the computation complexity of the MSDD decision statistic. This sub-

optimum algorithm achieves performance that is very close to that of the optimum algorithm. It can be shown that with an increasing observation interval, the performance of this kind of MSDD approaches that of OC with differential encoding.

The second MSDD scheme is developed for the case in which the only required channel information is the channel gain of the interference. It is shown that when the interference power level is high, this MSDD technique can achieve good performance.

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To my beloved family

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CHAPTER 1

INTRODUCTION

1.1 Motivation

In modern commercial wireless communication systems such as code division multiple access (CDMA) systems and time division multiple access (TDMA) systems, the cellular concept is widely applied to increase system capacity [1]. Under this concept, the entire service area is divided into small areas called cells. Several cells comprise a cell cluster. A cell cluster can use all the available frequency resources. For TDMA systems, each cell can use a portion of the available frequency channels, but neighboring cells use different frequency channels. The cells that use the same frequency channels are at least a cell away from each other. In this way, the limited precious frequency resource could be reused, hence the system capacity can be increased infinitely (at least in theory), while at the same time the interference is kept to a minimum. What should be kept in mind is that, though the interference is reduced, it still exists due to the fact that the same frequency channel is used by different cells. In fact, the performance of both TDMA and CDMA cellular systems is interference limited.

The cellular concept is shown in Figure 1.1. In the figure, A_i to G_i (i =0, 1, ..., 6) are cells that form a cell cluster. Cells A_i for i =0, 1, ..., 6 use the same frequency channels, so do cells B_i to G_i. The user s_i in cells A_i (i =1, 2, ..., 6) can interfere with the user s_0 in cell A_0.

Another factor that constrains the performance of wireless systems is multipath fading. In the wireless environment, due to reflection, diffraction and scattering, the transmitted signal may reach the receiving antenna through more than one path (shown in Figure 1.2) and fade greatly due to the interference between paths. Since

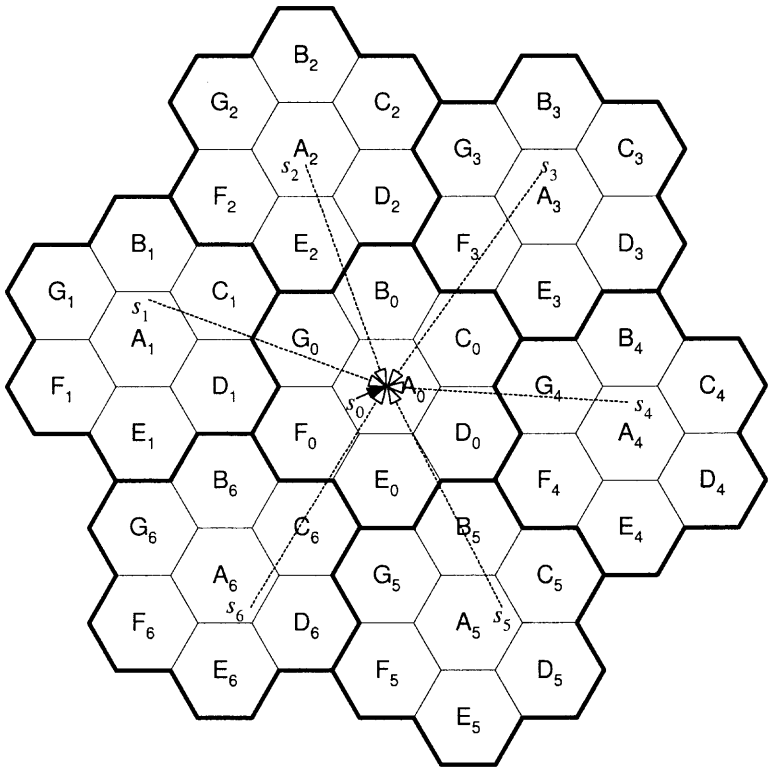

Figure 1.1 The cellular concept. Since cells A_i for $i = 1, ..., 6$ use the same frequency channels as cell A_0, the user s_i (i =1, 2, ..., 6) may interfere with the user s_0 in cell A_0.

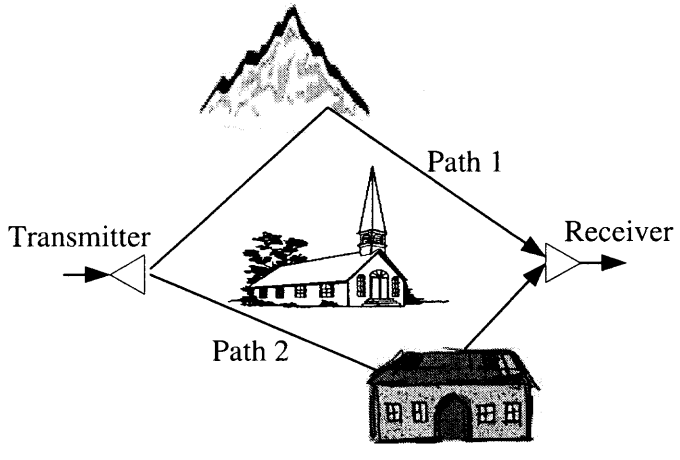

Figure 1.2 Multipath. The transmitted signal reaches the receiver through path 1 and path 2.

the locations of the transmitter, the obstacles, and the receiver are random, the transmitting paths are random as well. The total effect of the path's interference is a random attenuation of the transmitted signal. When the attenuation is deep, the received signal is so weak that the receiver cannot recover the transmitted signal. To resolve this problem, diversity is introduced. With diversity, several replicas of the same information signal are transmitted over independently fading channels. The probability that all the signal components reaching the receiver fade simultaneously is reduced considerably.

Three examples of diversity techniques are [2, Chapter 14]:

- Temporal diversity: the same information-bearing signal is transmitted in more than one time slot, where the separation between successive time slots equals or exceeds the coherence time of the channel.

- Frequency diversity: the same information-bearing signal is transmitted on more than one carrier frequency, where the separation between successive carrier frequencies equals or exceeds the coherence bandwidth of the channel.

- Spatial diversity: more than one transmitting and/or receiving antenna are employed. The antennas are spaced sufficiently far apart that the multipath components in the signal have independent fading.

Since spatial diversity does not require the expansion of bandwidth, it is desirable for bandwidth-limited systems when cost and size permit. And as pointed out in [3], spatial diversity could be used to cancel interference as well as to combat fading. Capacity of systems with spatial diversity has been proven to increase with the number of antennas [4].

It is for these advantages that the performance analysis of communication systems with spatial diversity has been an appealing research area. In practice receive diversity has been implemented at base stations. For example, in second generation

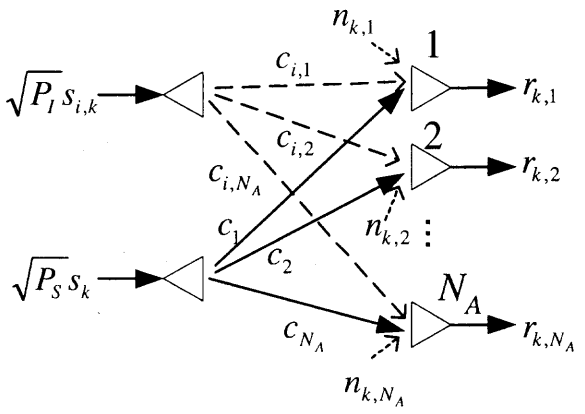

Figure 1.3 Diagram for systems with N_A receive diversity branches. s_k is the desired signal. There could be more than one interferer $s_{i,k}$ (only one is shown in the figure).

IS-136 TDMA [5], two receive antennas are deployed at base stations. Technology has been developed for deploying 4 receive antennas.

Currently much of the research on spatial diversity is focused on space-time codes ([6], [7], [8]), which employ transmit diversity. While space-time codes can provide some coding gain as well as spatial diversity, and could be the future application, this dissertation focuses on a more practical problem for now: performance analysis of communication systems with receive diversity. The basic system model used in this work is presented in the next section.

1.2 System Model

Consider a communication system with receive diversity but with a single transmit antenna. All the signals are represented as lowpass equivalents. As shown in Figure 1.3, there is one transmitting antenna, N_A receive branches, and N_I interferers in the system (only one is shown in the figure). The sampled output of the matched filter for the l-th branch at time k is expressed as

$$r_{k,l} = \sqrt{P_s}c_l s_k + \sum_{i=1}^{N_I} \sqrt{P_I}c_{i,l}s_{i,k} + n_{k,l},\ l = 1, \cdots, N_A \tag{1.1}$$

where the parameters in (1.1) are:

P_s : power of the desired signal.

c_l : channel gain of the l-th branch for the desired signal.

s_k : desired transmitted signal.

P_I : power of the interferers (assume all interferers have equal power).

$c_{i,l}$: channel gain of the l-th branch for the i-th interferer.

$s_{i,k}$: signal of the i-th interferer.

$n_{k,l}$: complex white Gaussian noise.

The signals s_k and $s_{i,k}$ could be multiple phase-shift keying (M-PSK) symbols, differential multiple phase-shift keying (M-DPSK) symbols, or Gaussian distributed signals. That will be defined more specifically in later chapters.

The received signal model in vector notation is

$$\mathbf{r}_k = \sqrt{P_s}\mathbf{c}s_k + \sqrt{P_I}\sum_{i=1}^{N_I}\mathbf{c}_i s_{i,k} + \mathbf{n}_k, \tag{1.2}$$

where $\mathbf{r}_k = [r_{k,1}, r_{k,2}, \cdots, r_{k,N_A}]^T$, and the superscript T denotes vector transposition; $\mathbf{c}, \mathbf{c}_i$ and $\mathbf{n}_k$ are vectors that are defined similarly to $\mathbf{r}_k$. For future use, the fading matrix for the interferers is also defined as $\mathbf{C}_I = [\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_{N_I}]$. The interference plus noise vector is defined as

$$\mathbf{z}_k = \sqrt{P_I}\sum_{i=1}^{N_I}\mathbf{c}_i s_{i,k} + \mathbf{n}_k. \tag{1.3}$$

The covariance matrix of $\mathbf{z}_k$ conditioned on $\mathbf{c}_i, i = 1, 2, \cdots, N_I$ is

$$\mathbf{R} = P_I\sum_{i=1}^{N_I}\mathbf{c}_i\mathbf{c}_i^H + \text{diag}\ \left(\sigma_1^2, \sigma_2^2, \cdots, \sigma_{N_A}^2\right) \tag{1.4}$$

where the superscript H denotes the Hermitian transposition and $\left(\sigma_1^2, \sigma_2^2, \cdots, \sigma_{N_A}^2\right)$ is the power profile of the noise.

1.3 Background

For wireless communication systems with receive diversity, optimum combining (OC) is a well-known approach to combat fading and suppress cochannel interference. The maximum-likelihood decision rule for OC is

$$\widehat{s}_k = \arg \max_{s_k} p\left(\mathbf{r}_k | s_k, \mathbf{c}, \mathbf{R}\right), \tag{1.5}$$

where $p\left(\mathbf{r}_k | s_k, \mathbf{c}, \mathbf{R}\right)$ is the probability of $\mathbf{r}_k$ conditioned on $s_k, \mathbf{c}$, and $\mathbf{R}$. A simplified version of this decision rule will be shown in Chapter 2 for BPSK modulation.

One of the efforts in this dissertation is to derive closed-form expressions for symbol error probability (SEP) and bit error probability (BEP) for OC. These kinds of expressions have been obtained before but only for some special cases. Some related work about OC is summarized in Chapter 2.

OC is a coherent detection scheme. To construct the weight vector $\mathbf{w}$, the following information is required: $\mathbf{c}$, the channel gain (amplitude and phase) of the desired signal, and $\mathbf{R}$, the covariance matrix of the interference plus noise. For communication systems where channel phase information is very difficult or impossible to recover, OC is not practical. Under this circumstance, a non-coherent detection scheme must be considered.

One such non-coherent scheme is differential detection of differentially encoded signals. For conventional differential detection, two received signals are used in the observation interval to make decisions about the transmitted signal. The recovery of the channel phase is not required. The decision rule for conventional differential detection is

$$(\widehat{s}_{k-1}, \widehat{s}_k) = \arg \max_{s_{k-1}, s_k} p\left(\mathbf{r}_{k-1}, \mathbf{r}_k | s_{k-1}, s_k, \mathbf{R}\right), \tag{1.6}$$

where $p\left(\mathbf{r}_{k-1}, \mathbf{r}_k | s_{k-1}, s_k, \mathbf{R}\right)$ is the probability of $\mathbf{r}_{k-1}, \mathbf{r}_k$ conditioned on s_{k-1}, s_k and $\mathbf{R}$. Conventional differential detection suffers a performance penalty compared to coherent detection.

Multiple-symbol differential detection (MSDD) achieves better performance than conventional differential detection (but not as good as coherent detection). In MSDD, more than two symbols are used in the observation interval. It was shown that with the increase of the number of symbols in the observation interval, the performance of differential detection can be improved significantly. The decision rule for MSDD is a generalization of (1.6)

$$\widehat{\mathbf{s}}_k = \arg\max_{\mathbf{s}_k} p\left(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{R}\right), \tag{1.7}$$

where $\mathbf{s}_k \triangleq \left[s_{k-(K-1)}, \cdots, s_{k-1}, s_k\right]^T$ is a sequence of K $(K \geqslant 2)$ symbols, $\underline{\mathbf{r}}_k = \left[\mathbf{r}_{k-(K-1)}, \cdots, \mathbf{r}_k\right]^T$ is a vector of all the received signals in the observation interval, and $p\left(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{R}\right)$ is the probability of $\underline{\mathbf{r}}_k$ conditioned on $\mathbf{s}_k$ and $\mathbf{R}$. Some related work about MSDD is summarized in Chapter 5. The channels are assumed to be static within the transmitted sequence of K symbols.

In this work, MSDD is applied to communication systems with interference. The only required channel information for that kind of MSDD is the covariance matrix of the interference plus noise $\mathbf{R}$. By simulation and analysis results, it is demonstrated that asymptotically with increasing observation block, MSDD achieves performance close to that of OC with differential encoding.

MSDD is also developed for another kind of non-coherent detection, where the only required channel information is the channel amplitude of the interference. For the case where there is only one interference source, the decision rule is

$$\left(\widehat{\mathbf{s}}_k, \widehat{\mathbf{s}}_{I,k}\right) = \arg\max_{\mathbf{s}_k, \mathbf{s}_{I,k}} p\left(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{s}_{I,k}, |\mathbf{c}_I|\right), \tag{1.8}$$

where $\mathbf{s}_k$ and $\underline{\mathbf{r}}_k$ were defined previously; $\mathbf{s}_{I,k} \triangleq [s_{I,k-(K-1)}, \cdots, s_{I,k-1}, s_{I,k}]^T$ is the sequence of interference symbols; $|\mathbf{c}_I|$ is short notation for the channel gain of the interference, and $p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{s}_{I,k}, |\mathbf{c}_I|)$ is the probability of $\underline{\mathbf{r}}_k$ conditioned on $\mathbf{s}_k$, $\mathbf{s}_{I,k}$, and $|\mathbf{c}_I|$.

1.4 Outline of the Dissertation

The main topics of this dissertation are:

1. Error probability analysis of OC.
2. Derivation of the decision statistic for MSDD and analysis of its performance.
3. Performance comparison of OC and MSDD.

The first topic is covered in Chapters 2 to 4, while the last two topics are covered in Chapters 5 and 6. The chapter outlines are as follows:

Chapter 2: The decision metric of OC is used to derive the closed-form expressions of bit error probability (BEP) for OC. The BEP conditioned on the fading of the interference is derived first, then the unconditional BEP is obtained. The expressions are for systems with binary phase-shift keying (BPSK) modulation, multiple interferers, and multiple receive branches.

Chapter 3: By using the moment generating function (MGF) of the output signal to interference plus noise ratio (SINR), expressions for both symbol error probability (SEP) and BEP for M-PSK modulation are derived. The final expressions involve only a single integration over elementary functions. With these expressions, it takes less time to evaluate the SEP and BEP than it would take to carry out Monte Carlo simulations or to evaluate multiple-fold integrals. Simple asymptotic expressions for BEP of OC for M-PSK modulation are also derived. Numerical results are used to show how close the asymptotic results are to the exact results.

Chapter 4: The probability density function (PDF) of the output SINR for OC, which can be obtained from the reliability function (defined as the probability that the

SINR is less than a threshold), is used to derive an expression for the SEP for M-PSK modulation. The final expression only involves a single integral with finite limits and finite integrand. A closed-form expression for the SEP of BPSK modulation is also derived. The new expressions for both M-PSK and BPSK are the first expressions that can be used to evaluate the exact SEP of systems with interferers of unequal power level.

Chapter 5: A detector exploiting MSDD technique is developed. The channel gain of the desired signal is assumed to be unknown. M-DPSK modulation is employed. The decision statistic for the detector is derived based on the principle of maximum-likelihood sequence detection (MLSD). The performance of the detector is demonstrated by simulation and analysis.

Chapter 6: Another kind of MSDD is presented to suppress cochannel interference. The channel gain of the desired signal and the channel phase of the interference are assumed to be unknown, but the channel amplitude of interference is assumed to be known at the receiver. The interference signal is assumed to have the same M-DPSK modulation as the desired signal. A maximum-likelihood sequence detector is developed for detecting both the desired signal and the interference signal. This receiver can be viewed as a kind of multiuser detector employing MSDD.

Summary and future work are presented in Chapter 7.

CHAPTER 2

BEP ANALYSIS FOR OC WITH BPSK MODULATION

2.1 Introduction

As mentioned in Chapter 1, for wireless communication systems with receive diversity, OC is an efficient approach to combat fading and suppress cochannel interference. It combines the output of the receive branches in an optimum way and achieves the maximum output SINR.

Performance analysis of OC has been an active research area. Analysis for the case of a single interference source can be found in [3, 9, 10]. In [3, 9], Rayleigh fading is assumed for the desired signal, but mean values, rather than actual distributions, are used to represent fading effects on the interference. In [10], exact expressions (requiring integration) are developed under the assumption of Rayleigh fading for both the desired signal and interference. Closed-form expressions of the BEP for this case were obtained in [11].

The case of multiple interferers is more challenging. Closed-form expressions of the BEP for a number of interferers no less than the number of receive branches and negligible thermal noise with BPSK modulation were developed in [12]. The performance of systems with multiple interferers has been studied extensively through Monte Carlo simulations [3], capacity [13], upper bound [14, 15, 16], approximate expressions [16, 17, 18], and exact expressions with integral forms [19, 20]. The performance of OC was compared with that of maximum ratio combining (MRC) in [21]. Performance of OC in the presence of channel correlations is evaluated in [22, 23]. A comprehensive treatment of diversity and OC methods can be found in [24].

The conventional way of deriving the expression for SEP or BEP often starts with the PDF or the MGF of the SINR as will be detailed in Chapters 3 and 4. These approaches yield closed-form expressions for the BEP of BPSK modulation [25]. In the present chapter, a different approach is taken by performing the analysis directly on the decision statistic rather than on the SINR. It is shown that, for BPSK, this approach allows exact BEP analysis and it requires averaging only over the fading of the interference. Although the algebra is somewhat cumbersome, at the end this method provides a closed-form expression.

This chapter is organized as follows: Following the system model in Section 2.2, the conditional BEP is derived in Section 2.3. In Section 2.4, the conditional BEP is averaged over the fading of the interference to get the unconditional BEP. Numerical results are presented in Section 2.5.

2.2 System Model

The system model used in this chapter is similar to that mentioned in Chapter 1, Section 1.2. For OC, symbol by symbol detection is performed and time does not affect the analysis. Hence the time index k in (1.1) can be dropped and the system model is rewritten as

$$r_l = \sqrt{P_s} c_l s + \sum_{i=1}^{N_I} \sqrt{P_I} c_{i,l} s_i + n_l, \; l = 1, 2, \cdots, N_A, \tag{2.1}$$

where all quantities are defined similarly to (1.1). The symbol s is assumed to be BPSK. The channel gains c_l and $c_{i,l}$ are assumed to be independent and identically distributed (i.i.d.), zero-mean, circularly symmetric, complex Gaussian random variables (Rayleigh fading), with variance 1/2 per dimension. The signal model in vector notation is

$$\mathbf{r} = \sqrt{P_s}\mathbf{c}s + \sqrt{P_I}\sum_{i=1}^{N_I} \mathbf{c}_i s_i + \mathbf{n} = \sqrt{P_s}\mathbf{c}s + \mathbf{z}, \tag{2.2}$$

where $\mathbf{r} = [r_1, r_2, \cdots, r_{N_A}]^T$, $\mathbf{c}$, $\mathbf{c}_i$, and $\mathbf{n}$ are defined similarly, $\mathbf{z} = \sqrt{P_I}\sum_{i=1}^{N_I} \mathbf{c}_i s_i + \mathbf{n}$ is the interference plus noise vector.

It is further assumed that conditioned on the vectors $\mathbf{c}_i$, the interference plus noise vector $\mathbf{z}$ has a multivariate complex-Gaussian distribution with zero mean and covariance matrix $\mathbf{R} = E\left[\mathbf{z}\mathbf{z}^H\right]$,

$$\mathbf{R} = P_I \sum_{i=1}^{N_I} \mathbf{c}_i \mathbf{c}_i^H + \sigma^2 \mathbf{I}_{N_A}, \tag{2.3}$$

where the superscript H denotes the Hermitian transposition, σ^2 is the power of the noise, and $\mathbf{I}_{N_A}$ is an identity matrix of rank N_A.

Define $N_{\max} = \max(N_A, N_I)$ and $N_{\min} = \min(N_A, N_I)$. Diagonalize $\mathbf{R}$ as $\mathbf{R} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^H$, where $\mathbf{\Lambda} = \text{diag}\ (\lambda_1, \lambda_2, \cdots, \lambda_{N_A})$, $\lambda_1, \lambda_2, \cdots, \lambda_{N_A}$ are the eigenvalues of $\mathbf{R}$ listed in descending order, and $\mathbf{U}$ is a unitary matrix whose columns are the eigenvectors of $\mathbf{R}$. Assume that the vectors $\mathbf{c}_i$ (for $i = 1, 2, \cdots, N_I$) are linearly independent (a reasonable assumption since the components of these vectors are realizations of mutually independent random variables). It follows that [20] $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{N_{\min}}$ are random variables, while $\lambda_m = \sigma^2$ for $m = N_{\min} + 1, N_{\min} + 2, \cdots, N_A$. For later use, denote the vector of non-trivial eigenvalues as $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \cdots, \lambda_{N_{\min}}]^T$. The inverse covariance matrix of $\mathbf{R}$ is $\mathbf{R}^{-1} = \mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{U}^H$.

2.3 Derivation of Conditional BEP

In this section and the next, the theoretical analysis of the BEP of OC for BPSK modulation is carried out.

As shown in Figure 2.1, for the OC detector, the components of the received signal vector $\mathbf{r}$ are weighted and combined to obtain the output signal. The weight vector that yields the maximum SINR is $\mathbf{w} = \mathbf{R}^{-1}\mathbf{c}$ [3][1]. The output of the combiner is $\mathbf{w}^H\mathbf{r}$. For BPSK modulation, the decision rule of the detector is: if $\text{Re}\left(\mathbf{w}^H\mathbf{r}\right) \geqslant 0$,

[1]In this report, $\mathbf{R}$ is the interference plus noise covariance matrix. Some authors compute the optimal combining weight vector from the signal plus interference and noise covariance

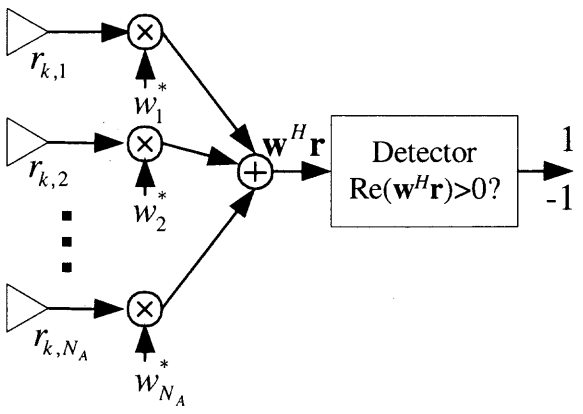

Figure 2.1 Diagram of the OC detector for BPSK modulation.

the decision is made that 1 is transmitted; otherwise the decision is made that -1 is transmitted. Due to the symmetry of the BPSK constellation and assuming a source with equal symbol probabilities , it suffices to analyze the case of $s = 1$. For this case the received signal is $\mathbf{r} = \sqrt{P_s}\mathbf{c} + \mathbf{z}$. Define

$$D = 2\mathrm{Re}\left(\mathbf{w}^H\mathbf{r}\right) = \mathbf{w}^H\mathbf{r} + \left(\mathbf{w}^H\mathbf{r}\right)^*, \tag{2.4}$$

where "*" denotes complex conjugation. According to the decision rule, when $D < 0$, the decision is made that -1 is transmitted and an error occurs. Therefore the BEP is $P_{\mathrm{b,BPSK}} = \Pr(D < 0)$. The analysis has two steps. First, the BEP is expressed conditioned on the fading of the interference. Subsequently, the conditioned BEP is averaged over the fading of the interference.

Fixing the values of the channels $\mathbf{c}_i$ of the interference sources leads to fixed values of the eigenvalues of the interference plus noise covariance matrix $\mathbf{R}$. These eigenvalues λ_m form the diagonal of the matrix $\mathbf{\Lambda}$. Substituting $\mathbf{w} = \mathbf{R}^{-1}\mathbf{c}$ into (2.4) and de-composing $\mathbf{R}^{-1}$ as $\mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{U}^H$, D can be expressed as

$$D = \sum_{m=1}^{N_A} \lambda_m^{-1}\left(g_m^* x_m + g_m x_m^*\right), \tag{2.5}$$

matrix. As shown in [26] and can be readily verified, the resulting weight vectors provide the same performance as they differ only by a scaling factor.

where λ_m's are the eigenvalues of $\mathbf{R}$ defined previously, x_m's are elements of the whitened observation vector

$$\mathbf{x} = [x_1, x_2, \cdots, x_{N_A}]^T = \mathbf{U}^H \mathbf{r}, \tag{2.6}$$

and g_m's are elements of the modified channel vector

$$\mathbf{g} = [g_1, g_2, \cdots, g_{N_A}]^T = \mathbf{U}^H \mathbf{c}. \tag{2.7}$$

Conditioned on the eigenvalues λ_m, the variable D is a quadratic form of Gaussian random variables. The goal is to evaluate the conditional BEP $P_{\text{b,BPSK}}(E|\boldsymbol{\lambda}) = \Pr(D < 0|\boldsymbol{\lambda})$, where the notation indicates the dependency on the $N_{\min}$ largest eigenvalues of $\mathbf{R}$ (the other $(N_A - N_{\min})$ eigenvalues are equal to the constant σ^2).

Let $\Phi_{D|\boldsymbol{\lambda}}(j\omega)$ be the characteristic function of D conditioned on $\boldsymbol{\lambda}$. Using results from [2, Appendix B], it can be shown that the conditional BEP is

$$P_{\text{b,BPSK}}(E|\boldsymbol{\lambda}) = -\frac{1}{2\pi j}\int_{-\infty+j\varepsilon}^{\infty+j\varepsilon} \frac{\Phi_{D|\boldsymbol{\lambda}}(j\omega)}{\omega} d\omega = -\sum_{\text{Im}(\omega_n)>0} \text{Res}\left[\frac{\Phi_{D|\boldsymbol{\lambda}}(j\omega)}{\omega}; \omega_n\right], \tag{2.8}$$

where ε is a small positive number and $\text{Res}\left[\Phi_{D|\boldsymbol{\lambda}}(j\omega)/\omega; \omega_n\right]$ denotes the residue of $\Phi_{D|\boldsymbol{\lambda}}(j\omega)/\omega$ at pole ω_n. The summation is taken over the poles in the upper half of the complex plane.

In (2.5), D is a quadratic form of complex-valued random vectors $\mathbf{x}$ and $\mathbf{g}$. By applying the results in [2, Appendix B] to (2.5) (see Appendix A for details), it can be shown that the characteristic function $\Phi_{D|\boldsymbol{\lambda}}(j\omega)$ is

$$\Phi_{D|\boldsymbol{\lambda}}(j\omega) = \left[\frac{\omega_1\omega_2}{(\omega-\omega_1)(\omega-\omega_2)}\right]^{N_A-N_{\min}} \prod_{m=1}^{N_{\min}} \frac{\omega_{1,m}\omega_{2,m}}{(\omega-\omega_{1,m})(\omega-\omega_{2,m})}, \tag{2.9}$$

where

$$\omega_1 = -j\left(\sqrt{P_s+\sigma^2}-\sqrt{P_s}\right) \tag{2.10}$$

$$\omega_2 = j\left(\sqrt{P_s+\sigma^2}+\sqrt{P_s}\right) \tag{2.11}$$

$$\omega_{1,m} = -j\left(\sqrt{P_s+\lambda_m}-\sqrt{P_s}\right) \tag{2.12}$$

$$\omega_{2,m} = j\left(\sqrt{P_s+\lambda_m}+\sqrt{P_s}\right). \tag{2.13}$$

The residue in (2.8) is evaluated in Appendix B and obtained as (B.13). After substituting $\omega_1, \omega_2, \omega_{1,m}$, and $\omega_{2,m}$ into (B.13), $P_{\text{b,BPSK}}(E|\boldsymbol{\lambda})$ can be obtained as

$$\begin{aligned}
&P_{\text{b,BPSK}}(E|\boldsymbol{\lambda}) \\
&= (-1)^{N_A+1}\sum_{m=1}^{N_{\min}}\frac{\lambda_m}{2\sqrt{P_s+\lambda_m}\left(\sqrt{P_s+\lambda_m}+\sqrt{P_s}\right)}\frac{\left(\sigma^2\right)^{N_A-N_{\min}}}{\left(\lambda_m-\sigma^2\right)^{N_A-N_{\min}}} \\
&\times\prod_{n=1,n\neq m}^{N_{\min}}\frac{\lambda_n}{(\lambda_m-\lambda_n)}+\left(\sigma^2\right)^{N_A-N_{\min}}\sum_{l=0}^{N_A-N_{\min}-1}\binom{N_A-N_{\min}+l-1}{l} \\
&\times\left\{1+\frac{1}{2}(-1)^{N_{\min}}\left(\sqrt{P_s+\sigma^2}+\sqrt{P_s}\right)^{N_A-N_{\min}-l}\left(\prod_{n=1}^{N_{\min}}\lambda_n\right)\right. \\
&\times\sum_{m=1}^{N_{\min}}\left(\prod_{n=1,n\neq m}^{N_{\min}}\frac{1}{\lambda_m-\lambda_n}\right)\frac{1}{\sqrt{P_s+\lambda_m}} \\
&\times\left[\frac{1}{\left(\sqrt{P_s+\lambda_m}-\sqrt{P_s}\right)}\frac{1}{\left(\sqrt{P_s+\sigma^2}+\sqrt{P_s+\lambda_m}\right)^{N_A-N_{\min}-l}}\right. \\
&\left.\left.+\frac{1}{\left(\sqrt{P_s+\lambda_m}+\sqrt{P_s}\right)}\frac{1}{\left(\sqrt{P_s+\sigma^2}-\sqrt{P_s+\lambda_m}\right)^{N_A-N_{\min}-l}}\right]\right\} \\
&\times\frac{1}{\left(\sqrt{P_s+\sigma^2}+\sqrt{P_s}\right)^{N_A-N_{\min}-l}\left(2\sqrt{P_s+\sigma^2}\right)^{N_A-N_{\min}+l}}.
\end{aligned} \tag{2.14, 2.15}$$

When the number of receive branches N_A is less than or equal to the number of interferers, i.e., $N_A \leq N_I$, $N_{\min} = N_A$, the summation $\sum_{l=0}^{N_A-N_{\min}-1}$ in (2.14) is equal to zero.

As an example, the expression of BEP for the special case of no interference is now derived.

2.3.1 Special Case: No Interference

In this case, $N_{\min} = N_I = 0$, OC becomes MRC. Eq. (2.9) becomes

$$\Phi_D(j\omega) = \left[\frac{\omega_1\omega_2}{(\omega-\omega_1)(\omega-\omega_2)}\right]^{N_A-N_{\min}}. \tag{2.16}$$

Substitute the above expression in (2.8), and carry out the calculation of the residue. Then

$$P_{\text{b,BPSK}} = \left[\frac{1}{2}(1-\mu)\right]^{N_A}\left\{\sum_{k=0}^{N_A-1}\binom{N_A-1+k}{k}\left[\frac{1}{2}(1+\mu)\right]^k\right\}, \tag{2.17}$$

where $\mu = \sqrt{\gamma/(1+\gamma)}$, and

$$\gamma = \frac{P_s}{\sigma^2} \tag{2.18}$$

is the signal to noise ratio (SNR). Since there is no interference, the BEP is unconditional. (2.17) is the same as (14-4-15) in [2]. Note that the derivation of (2.17) did not require integration. In [2], the BEP of MRC is obtained by integration. It was shown that [2, Eq. (14-4-18)] for SNR $\gamma \gg 1$,

$$P_{\text{b,BPSK}} \approx \binom{2N_A-1}{N_A}\frac{1}{(4\gamma)^{N_A}}. \tag{2.19}$$

Eq. (2.19) will be used later for comparison.

2.4 Derivation of Unconditional BEP

For the general case with interference, the unconditional BEP $P_{\text{b,BPSK}}$ is obtained by averaging the conditional BEP $P_{\text{b,BPSK}}(E|\boldsymbol{\lambda})$ over the fading of the interference $\mathbf{c}_i$, or equivalently over the eigenvalues $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \cdots, \lambda_{N_{\min}}]^T$,

$$P_{\text{b,BPSK}} = \int P_{\text{b,BPSK}}(E|\boldsymbol{\lambda}) p_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) d\boldsymbol{\lambda}, \tag{2.20}$$

where $p_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})$ is the joint probability density function (PDF) of the eigenvalues. Serendipitously, the PDF $p_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})$ was developed in [20] for a signal model similar to ours and is given

by

$$p_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) = K_0 \frac{1}{P_I^{N_{\min}}} \left[\prod_{i=1}^{N_{\min}} \exp\left(-\frac{\lambda_i - \sigma^2}{P_I}\right) \left(\frac{\lambda_i - \sigma^2}{P_I}\right)^{N_{\max}-N_{\min}} \right] \times \left[\prod_{1 \le i < j \le N_{\min}} \left(\frac{\lambda_i - \sigma^2}{P_I} - \frac{\lambda_j - \sigma^2}{P_I}\right)^2 \right], \tag{2.21}$$

where

$$K_0 = \frac{1}{\left[\prod_{i=1}^{N_{\min}} (N_{\max} - i)!\right] \left[\prod_{i=1}^{N_{\min}} (N_{\min} - i)!\right]}. \tag{2.22}$$

The conditional BEP in (2.15) is a non-rational function of the eigenvalues λ_m's. To facilitate the integration in (2.20), define the following transformation of variables

$$y_m = \sqrt{\frac{\lambda_m}{P_s} + 1}, \qquad m = 1, 2, \cdots, N_{\min} \tag{2.23}$$

and define the set $\mathbf{y} = [y_1, y_2, \cdots, y_{N_{\min}}]^T$. Since λ_m is random, y_m is random as well. Also define

$$\eta = \sqrt{\frac{\sigma^2}{P_s} + 1} = \sqrt{\frac{1}{\gamma} + 1}. \tag{2.24}$$

Then

$$\lambda_m = P_s\left(y_m^2 - 1\right) \quad m = 1, 2, \cdots, N_{\min} \tag{2.25}$$

$$\sigma^2 = P_s\left(\eta^2 - 1\right). \tag{2.26}$$

By substituting (2.25) and (2.26) into (2.15), and after some straightforward manipulations, the conditional BEP as a function of the variables y_m's is obtained as

$$P_{\text{b,BPSK}}(E|\mathbf{y}) = -\sum_{m=1}^{N_{\min}} f_m(\mathbf{y}) + (-1)^{N_A - N_{\min}} \sum_{l=0}^{N_A - N_{\min} - 1} \binom{N_A - N_{\min} + l - 1}{l} \times \left[1 + \sum_{m=1}^{N_{\min}} h_{m,l}(\mathbf{y})\right] \frac{(1-\eta)^{N_A - N_{\min} - l}}{(2\eta)^{N_A - N_{\min} + l}} \left(-\frac{1}{\gamma}\right)^l, \tag{2.27}$$

where the functions $f_m(\mathbf{y})$ and $h_{m,l}(\mathbf{y})$ are defined respectively as

$$f_m(\mathbf{y}) = \frac{1-y_m}{2y_m} \frac{(1-\eta^2)^{N_A-N_{\min}}}{(y_m^2-\eta^2)^{N_A-N_{\min}}} \prod_{n=1,n\neq m}^{N_{\min}} \frac{1-y_n^2}{y_m^2-y_n^2} \tag{2.28}$$

and

$$\begin{aligned} h_{m,l}(\mathbf{y}) &= (-1)^{N_A-N_{\min}-l} \frac{(1+\eta)^{N_A-N_{\min}-l}}{2y_m} \frac{1}{(y_m^2-\eta^2)^{N_A-N_{\min}-l}} \\ &\quad \times b_{N_A-N_{\min}-l}(y_m) \prod_{n=1,n\neq m}^{N_{\min}} \frac{1-y_n^2}{y_m^2-y_n^2}. \end{aligned} \tag{2.29}$$

The function $b_k(y_m)$ in (2.29) is in turn defined for $1 \leq k \leq N_A - N_{\min}$ as

$$b_k(y_m) = -(1+y_m)(\eta-y_m)^k + (1-y_m)(\eta+y_m)^k. \tag{2.30}$$

Clearly, the conditional BEP $P_{\text{b,BPSK}}(E|\mathbf{y})$ is a rational function of the elements of the set $\mathbf{y}$. By using the Jacobian of the transformation from $\underline{\boldsymbol{\lambda}}$ to $\mathbf{y}$, the joint PDF of $\mathbf{y}$ is

$$\begin{aligned} p_{\mathbf{y}}(\mathbf{y}) &= K_1 \left\{ \prod_{i=1}^{N_{\min}} \exp\left[-\beta\left(y_i^2-\eta^2\right)\right] \left(y_i^2-\eta^2\right)^{N_{\max}-N_{\min}} \right\} \\ &\quad \times \left[\prod_{1\leq i<j\leq N_{\min}} \left(y_i^2-y_j^2\right)^2 \right] y_1 y_2 \cdots y_{N_{\min}} \end{aligned} \tag{2.31}$$

for $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_{N_{\min}} \geqslant \eta$, where

$$\beta = \frac{P_s}{P_I} \tag{2.32}$$

is the signal to interference ratio (SIR) and

$$K_1 = \frac{2^{N_{\min}}}{\left[\prod_{i=1}^{N_{\min}} (N_{\max}-i)!\right]\left[\prod_{i=1}^{N_{\min}} (N_{\min}-i)!\right]} \beta^{N_{\max}N_{\min}}. \tag{2.33}$$

The unconditional BEP $P_{\text{b,BPSK}}$ is obtained by averaging the conditional $P_{\text{b,BPSK}}(E|\mathbf{y})$ over the random variables in the set $\mathbf{y}$,

$$\begin{aligned} & P_{\text{b,BPSK}} \\ = & \int P_{\text{b,BPSK}}(E|\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\ = & -\sum_{m=1}^{N_{\min}} \int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} + (-1)^{N_A - N_{\min}} \sum_{l=0}^{N_A - N_{\min} - 1} \binom{N_A - N_{\min} + l - 1}{l} \\ & \times \left(-\frac{1}{\gamma}\right)^l \left[1 + \sum_{m=1}^{N_{\min}} \int h_{m,l}(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y}\right] \frac{(1-\eta)^{N_A - N_{\min} - l}}{(2\eta)^{N_A - N_{\min} + l}}. \end{aligned} \tag{2.34}$$

This expression can be used for any number of diversity branches N_A and any number of interferers N_I. Since, as previously mentioned, $N_A \leq N_I$, $N_{\min} = N_A$, the summation $\sum_{l=0}^{N_A - N_{\min} - 1}$ is equal to zero; therefore, only the first term $-\sum_{m=1}^{N_{\min}} \int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y}$ is required to calculate the BEP for $N_A \leq N_I$.

Next the terms of (2.34) are evaluated.

2.4.1 Evaluation of $\sum_{m=1}^{N_{\min}} \int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y}$

The following definitions are needed:

1. B_q is a sequence defined as

$$B_q = \begin{cases} \sqrt{\frac{\pi}{\beta}} \exp(\beta\eta^2)\, Q\left(\sqrt{2\beta}\eta\right) & q = 0 \\ \frac{\eta}{2\beta} + \left(\frac{1}{2\beta} - \eta^2\right) \sqrt{\frac{\pi}{\beta}} \exp(\beta\eta^2)\, Q\left(\sqrt{2\beta}\eta\right) & q = 1 \\ \left(\frac{2q-1}{2\beta} - \eta^2\right) B_{q-1} + \frac{q-1}{\beta} \eta^2 B_{q-2} & q \geq 2 \end{cases}, \tag{2.35}$$

where $Q(\cdot)$ is the Gaussian Q-function. Note that the values of B_q for $q \geq 2$ can be evaluated recursively from the values of B_{q-1} and B_{q-2}. Since (2.35) is a second order difference equation with initial values for B_q, by using the method detailed in [27], this equation can be solved and B_q (for $q \geq 2$) can be expressed

in terms of the initial values B_0 and B_1 as:

$$B_q = v_{q-1,1}B_1 + v_{q-1,2}B_0, \tag{2.36}$$

where

$$v_{q,j} = \sum_{r=1}^{q+j-1} \sum_{\substack{l_1+\cdots+l_r=q+j-1 \\ l_i\in\{1,2\},l_r\geq j}} \left[\prod_{m=1}^{r} e_{q-\sum_{n=1}^{m-1} l_n, l_m}\right] \tag{2.37}$$

and

$$e_{q,1} = \frac{2q+1}{2\beta} - \eta^2 \tag{2.38}$$

$$e_{q,2} = \frac{q\eta^2}{\beta}. \tag{2.39}$$

The second summation in (2.37) is taken over all sets of indices satisfying the stated conditions. Substituting B_0 and B_1 in (2.36), then B_q (for $q \geq 2$) in closed-form is:

$$B_q = v_{q-1,1}\frac{\eta}{2\beta} + \left[\left(\frac{1}{2\beta} - \eta^2\right) v_{q-1,1} + v_{q-1,2}\right] \sqrt{\frac{\pi}{\beta}} \exp\left(\beta\eta^2\right) Q\left(\sqrt{2\beta}\eta\right). \tag{2.40}$$

2. $H_{p,q}$ is a function of integers p and q. For $0 \leq p, q \leq N_{\min} - 1$,

$$H_{p,q} = \frac{1}{\left[\prod_{i=1}^{N_{\min}} (N_{\max} - i)!\right] \left[\prod_{i=1}^{N_{\min}} (N_{\min} - i)!\right]} \times \sum_{\substack{m_1+\cdots+m_{N_{\min}-1}=N_{\min}-1-p \\ m_i\in\{0,1\}}} \sum_{\substack{n_1+\cdots+n_{N_{\min}-1}=N_{\min}-1-q \\ n_i\in\{0,1\}}} \det \mathbf{W}, \tag{2.41}$$

where for $N_{\min} = 1, \det \mathbf{W} = 1$; for $N_{\min} > 1$, $\det \mathbf{W}$ is the determinant of an $(N_{\min} - 1) \times (N_{\min} - 1)$ matrix whose i-th row, j-th column element is

$$W_{i,j} = (m_j + n_j + N_{\max} - N_{\min} + i + j - 2)!.$$

Using these definitions, in Appendix C it is shown that

$$\begin{aligned}
&\sum_{m=1}^{N_{\min}} \int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\
&= \left(-\frac{1}{\gamma}\right)^{N_A - N_{\min}} \beta^{N_{\max} - N_{\min} + 1} \sum_{p=0}^{N_{\min}-1} \sum_{q=0}^{N_{\min}-1} (-1)^q H_{p,q} \\
&\quad \times \frac{1}{\gamma^p} \left(B_{N_{\max} - N + q} - \frac{1}{2} \frac{(N_{\max} - N_A + q)!}{\beta^{N_{\max} - N_A + q + 1}} \right) \beta^{p+q}.
\end{aligned} \tag{2.42}$$

2.4.2 Evaluation of $\sum_{m=1}^{N_{\min}} \int h_{m,l}(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y}$

The function $b_k(y_m)$ defined in (2.30) can be alternatively expressed as (proven in Appendix E)

$$b_k(y_m) = 2y_m \sum_{t=0}^{[k/2]} a_{k,t} \left(y_m^2 - \eta^2\right)^t, \tag{2.43}$$

where $[k/2]$ denotes the largest integer that is equal to or less than $k/2$, and $a_{k,t}$ is evaluated as:

$$a_{k,t} = \left[\binom{k-1-t}{t} (1-\eta) - 2\eta \binom{k-1-t}{t-1} \right] (2\eta)^{k-1-2t}. \tag{2.44}$$

When calculating $a_{k,t}$, it is assumed that $\binom{m}{n} = 0$ for $m < n$ or $n < 0$.

Substituting (2.43) in (2.29) and using steps similar to those in Appendix C, then:

$$\begin{aligned}
&\sum_{m=1}^{N_{\min}} \int h_{m,l}(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\
&= (-1)^{N_A - N_{\min} - l} (1+\eta)^{N_A - N_{\min} - l} \beta^{N_A - N_{\min} + 1} \sum_{t=0}^{[(N_A - N_{\min} - l)/2]} a_{N_A - N_{\min} - l, t} \frac{1}{\beta^{l+t+1}} \\
&\quad \times \sum_{p=0}^{N_{\min}-1} \sum_{q=0}^{N_{\min}-1} (-1)^q (N_{\max} - N_A + l + t + q)! H_{p,q} \left(\frac{\beta}{\gamma}\right)^p.
\end{aligned} \tag{2.45}$$

Using the expressions obtained in (2.42) and (2.45), (2.34) can be evaluated to obtain the exact BEP for any given number of diversity branches N_A, number of interferers N_I, SNR $\gamma = P_s/\sigma^2$ and SIR $\beta = P_s/P_I$.

A simpler expression can be derived for the special case of $N_A > N_I$, SNR $\gamma \gg 1$ and SIR $\beta \ll 1$.

Special Case: $N_A > N_I$, SNR $\gamma \gg 1$ and SIR $\beta \ll 1$

In this case, $N_{\max} = N_A$, $N_{\min} = N_I$. Both (2.42) and (2.45) contain the term $\beta^{N_A - N_{\min}+1}$. Since $\beta \ll 1$ and $N_A > N_{\min}$, terms containing $\beta^{N_A - N_{\min}+1}$ are neglected. Therefore $\sum_{m=1}^{N_{\min}} \int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \approx 0$ and $\sum_{m=1}^{N_{\min}} \int h_{m,l}(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \approx 0$. By substituting these approximations in (2.34) and using $\eta = \sqrt{1/\gamma + 1} \approx 1 + 1/(2\gamma)$ for $\gamma \gg 1$,

$$P_{\text{b,BPSK}} \approx \binom{2(N_A - N_I) - 1}{(N_A - N_I)} \frac{1}{(4\gamma)^{N_A - N_I}}. \tag{2.46}$$

Comparing (2.46) with (2.19), it can be seen that for SNR $\gamma \gg 1$, the BEP of a system with N_A diversity branches and N_I ($N_I < N_A$) large interferers is equivalent to that of a system with $(N_A - N_I)$ diversity branches but without interference. This is a well-known result for OC [13].

2.5 Numerical Results

Figures 2.2 to 2.5 show the BEP versus SNR for different SIR β. Figures 2.2 to 2.4 are for $N_A = 4$ diversity branches, and $N_I = 1$, 2, 3 interferers, respectively. Figure 2.5 is for $N_A = 8$ diversity branches and $N_I = 5$ interferers. Figure 2.6 is for 4 branches, varying number of interferers, and SIR = 10.

In Figures 2.2, 2.4, 2.5 and 2.6, the interference generated in the simulations had a Gaussian distribution as assumed in developing the BEP analysis. Simulation results in Figure 2.3 were generated for two interference sources transmitting BPSK symbols. Analytical results were calculated using (2.34) and the related expressions such as (2.42) and (2.45).

In all the figures, the analysis results match the simulation results. This provides convincing demonstration of the validity of the analytical expression for BEP.

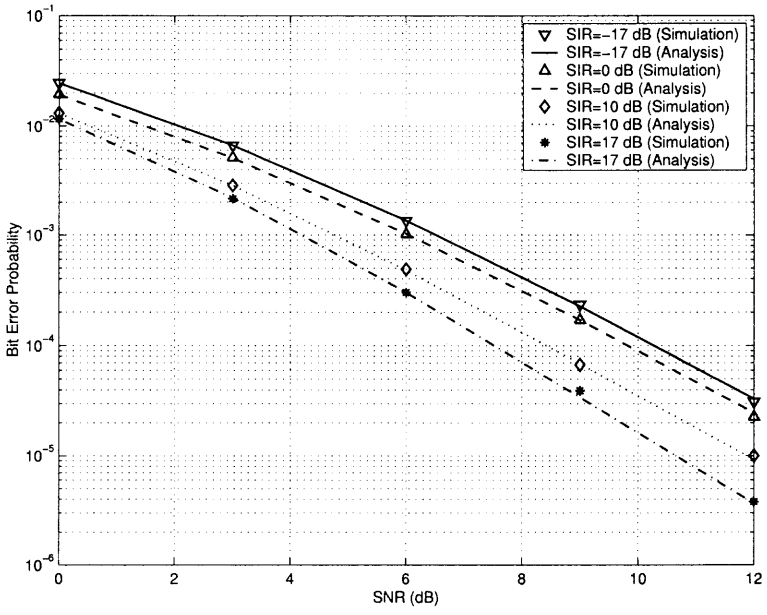

Figure 2.2 BEP versus SNR for $N_A = 4$ branches, $N_I = 1$ Gaussian distributed interferer.

As shown in Figure 2.3 for BPSK interference, the Gaussian assumption for the interference, while necessary for obtaining the theoretical results, is not critical for the accuracy of the BEP expressions. This can be explained by recognizing that the system has a sufficient number of degrees of freedom to suppress the interference sources effectively. The interference suppression is not sensitive to the Gaussian assumption. In fact, it is well known that OC maximizes the SINR irrespective of the density function governing the interference.

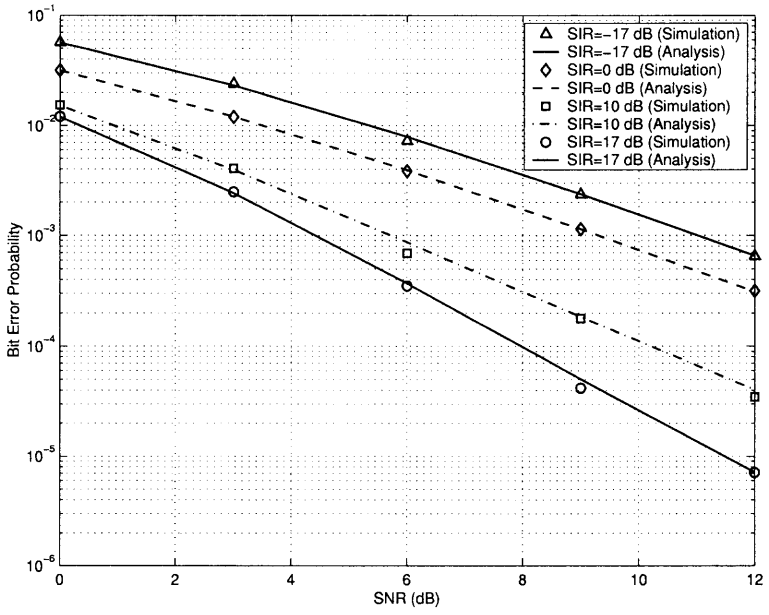

Figure 2.3 BEP versus SNR for $N_A = 4$ branches, $N_I = 2$ BPSK interferers.

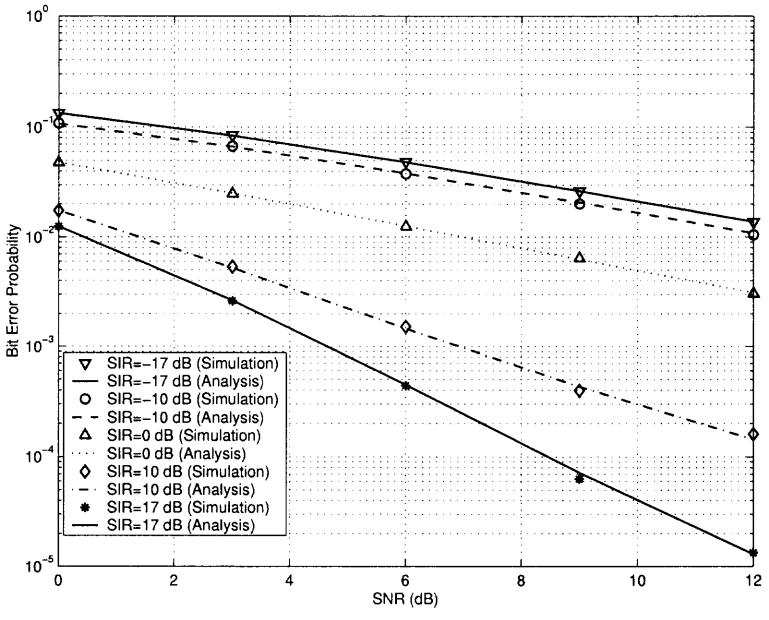

Figure 2.4 BEP versus SNR for $N_A = 4$ branches, $N_I = 3$ Gaussian distributed interferers.

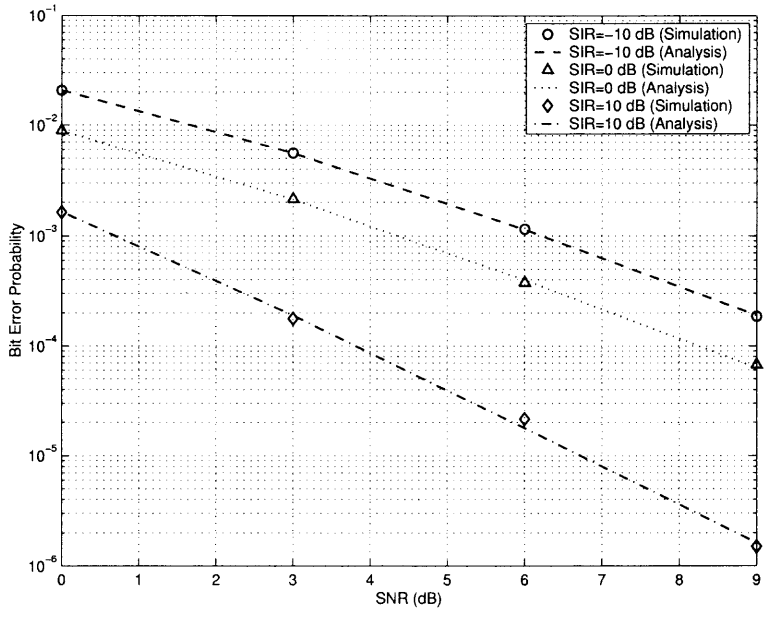

Figure 2.5 BEP versus SNR for $N_A = 8$ branches, $N_I = 5$ Gaussian distributed interferers.

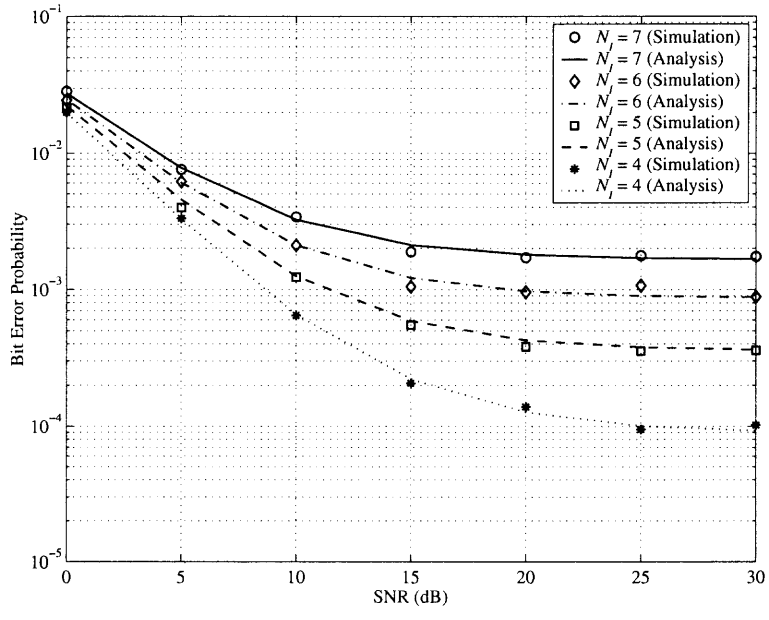

Figure 2.6 BEP versus SNR for $N_A = 4$ branches, SIR $= 10$. The number of interferers varies from $N_I = 4$ to $N_I = 7$.

CHAPTER 3

SEP AND BEP FOR OC WITH M-PSK MODULATION

3.1 Introduction

In Chapter 2, a new method was introduced for deriving the closed-form expression for the exact BEP for OC with BPSK modulation. The method started from the decision statistics of OC. This approach is not applicable to systems with M-PSK modulation.

An expression for SEP for M-PSK was derived in [20]. The expression is exact, and it applies to any number of interferers and receive branches. It involves $(N_{\min}+1)$-fold integration, where $N_{\min}$ is the minimum number of receive branches or interferers. An effective technique was derived to evaluated the SEP in [28]. A simpler and more elegant SEP expression was derived in recent work [29] for the same case. The expression contains integration over an integrand, which incorporates the incomplete Gamma function, itself an integral form.

In this chapter, expressions for both SEP and BEP for M-PSK are derived, with any number of receive branches and interferers. The moment generating function approach is taken to reach the final expressions, which involve only a single integration over elementary functions. With these expressions, it takes much less time to evaluate the SEP and BEP than it would take to carry out Monte Carlo simulations or to evaluate a multiple-fold integral.

The system model and assumption for this chapter are the same as those described in Chapter 2, with the exception that now the desired signal s is an M-PSK symbol. The expressions for SEP and BEP are developed in Section 3.2. Numerical results are shown in Section 3.3.

3.2 Expressions for SEP and BEP

With the OC detector, the components of the received signal vector $\mathbf{r}$ are weighted and combined to obtain the output signal. The weight vector yielding the maximum SINR is $\mathbf{w} = \mathbf{R}^{-1}\mathbf{c}$. The output of the combiner is

$$\mathbf{w}^H\mathbf{r} = \sqrt{P_s}\mathbf{c}^H\mathbf{R}^{-1}\mathbf{c}s + \mathbf{c}^H\mathbf{R}^{-1}\mathbf{z}. \tag{3.1}$$

The terms $\sqrt{P_s}\mathbf{c}^H\mathbf{R}^{-1}\mathbf{c}s$ and $\mathbf{c}^H\mathbf{R}^{-1}\mathbf{z}$ represent respectively, the desired signal and interference plus noise. The latter is Gaussian distributed conditioned on the channel vectors $\mathbf{c}$ and $\mathbf{c}_i$. The signal model of (3.1) is similar to that of an AWGN channel with noise variance $E_{s_i,\mathbf{n}}\left[\left|\mathbf{c}^H\mathbf{R}^{-1}\mathbf{z}\right|^2\right]$, with the expectation taken over the interfering signal s_i and AWGN $\mathbf{n}$.

3.2.1 Expression for SEP

For M-PSK signals over the AWGN channel, the SEP $P_{\mathrm{s},M\text{-PSK}}(E|\gamma)$ (conditioned on the SNR γ) can be expressed as [24, Eq. (8.22)]

$$P_{\mathrm{s},M\text{-PSK}}(E|\gamma) = \frac{1}{\pi}\int_0^{(M-1)\pi/M} \exp\left\{-\gamma\frac{\sin^2(\pi/M)}{\sin^2\theta}\right\} d\theta, \tag{3.2}$$

where M is the number of symbols of the M-PSK modulation, and γ is the symbol SNR. Likewise, for OC with M-PSK, the SEP can be written as

$$P_{\mathrm{s},M\text{-PSK}}(E|\gamma_t) = \frac{1}{\pi}\int_0^{(M-1)\pi/M} \exp\left\{-\gamma_t\frac{\sin^2(\pi/M)}{\sin^2\theta}\right\} d\theta, \tag{3.3}$$

where γ_t is the SINR at the output of the optimum combiner. The SEP is conditioned on channel realizations through γ_t. In order to get the ensemble average SEP $P_{\mathrm{s},M\text{-PSK}}$ for OC, $P_{\mathrm{s},M\text{-PSK}}(E|\gamma_t)$ has to be averaged over the distribution of γ_t,

$$P_{\mathrm{s},M\text{-PSK}} = \int_0^{\infty} P_{\mathrm{s},M\text{-PSK}}(E|\gamma_t)\, p_{\gamma_t}(\gamma_t)\, d\gamma_t, \tag{3.4}$$

where $p_{\gamma_t}(\gamma_t)$ is the PDF of the SINR γ_t. Let $p_{\gamma_t|\boldsymbol{\lambda}}(\gamma_t|\boldsymbol{\lambda})$ represent the PDF of γ_t conditioned on the non-trivial eigenvalues $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \cdots, \lambda_{N_{\min}}]^T$. The unconditional PDF $p_{\gamma_t}(\gamma_t)$ can be obtained by averaging $p_{\gamma_t|\boldsymbol{\lambda}}(\gamma_t|\boldsymbol{\lambda})$ over $\boldsymbol{\lambda}$:

$$p_{\gamma_t}(\gamma_t) = \int p_{\gamma_t|\boldsymbol{\lambda}}(\gamma_t|\boldsymbol{\lambda})\, p_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) d\boldsymbol{\lambda}. \tag{3.5}$$

By substituting (3.3) and (3.5) in (3.4), and after some manipulations similar to those in [24], it follows that

$$P_{\text{s},M\text{-PSK}} = \frac{1}{\pi} \int \left[\int_0^{(M-1)\pi/M} M_{\gamma_t|\boldsymbol{\lambda}} \left(-\frac{\sin^2(\pi/M)}{\sin^2\theta} \right) d\theta \right] p_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) d\boldsymbol{\lambda}, \tag{3.6}$$

where $M_{\gamma_t|\boldsymbol{\lambda}}(\cdot)$ is the MGF of the SINR γ_t conditioned on eigenvalues $\boldsymbol{\lambda}$. For the Rayleigh fading channel, the MGF is given by [24, Eq. 10.52]

$$M_{\gamma_t|\boldsymbol{\lambda}}(s) = \left(\frac{1}{1-\gamma s} \right)^{N_A - N_{\min}} \prod_{i=1}^{N_{\min}} \frac{1}{1 - \frac{P_s}{\lambda_i} s}, \tag{3.7}$$

where

$$\gamma = \frac{P_s}{\sigma^2}$$

is the symbol SNR.

3.2.2 Expression for BEP

For M-PSK modulation with Gray code bit mapping over the AWGN channel, the BEP $P_{\mathrm{b},M\text{-PSK}}(E|\gamma)$ is ([30], [24, Eq. (8.30)])

$$
\begin{aligned}
&P_{\mathrm{b},M\text{-PSK}}(E|\gamma) \\
&= \begin{cases}
P_0(E|\gamma) & M=2 \\
\frac{1}{2}\left[P_1(E|\gamma)+2P_2(E|\gamma)+P_3(E|\gamma)\right] & M=4 \\
\frac{1}{3}\left[P_1(E|\gamma)+2P_2(E|\gamma)+P_3(E|\gamma)+2P_4(E|\gamma)+\right. & \\
\left.3P_5(E|\gamma)+2P_6(E|\gamma)+P_7(E|\gamma)\right] & M=8 \\
\frac{1}{2}\left[\sum_{k=1}^{8}P_k(E|\gamma)+\sum_{k=2}^{5}P_k(E|\gamma)+\right. & \\
\left.P_5(E|\gamma)+2P_6(E|\gamma)+P_7(E|\gamma)\right] & M=16
\end{cases},
\end{aligned}
\tag{3.8}
$$

where

$$
\begin{aligned}
P_k(E|\gamma) &= \frac{1}{2\pi}\int_0^{\pi[1-(2k-1)/M]} \exp\left\{-\gamma\frac{\sin^2\left[(2k-1)\pi/M\right]}{\sin^2\theta}\right\}d\theta \\
&\quad -\frac{1}{2\pi}\int_0^{\pi[1-(2k+1)/M]} \exp\left\{-\gamma\frac{\sin^2\left[(2k+1)\pi/M\right]}{\sin^2\theta}\right\}d\theta.
\end{aligned}
\tag{3.9}
$$

For $M \geq 32$, similar expressions can be obtained [30].

Adapt these expressions for OC by averaging $P_k(E|\gamma)$ over γ_t (similar to the derivations from (3.2) to (3.6)) so that,

$$
\begin{aligned}
&P_k \\
&= \frac{1}{2\pi}\int\left\{\int_0^{\pi[1-(2k-1)/M]} M_{\gamma_t|\boldsymbol{\lambda}}\left(-\frac{\sin^2\left[(2k-1)\pi/M\right]}{\sin^2\theta}\right)d\theta\right\}p_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})d\boldsymbol{\lambda} \\
&\quad -\frac{1}{2\pi}\int\left\{\int_0^{\pi[1-(2k+1)/M]} M_{\gamma_t|\boldsymbol{\lambda}}\left(-\frac{\sin^2\left[(2k+1)\pi/M\right]}{\sin^2\theta}\right)d\theta\right\} \\
&\quad \times p_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})d\boldsymbol{\lambda}.
\end{aligned}
\tag{3.10}
$$

and

$$P_{\text{b},M\text{-PSK}} = \begin{cases} P_0 & M = 2 \\ \frac{1}{2}\left(P_1 + 2P_2 + P_3\right) & M = 4 \\ \frac{1}{3}\left(P_1 + 2P_2 + P_3 + 2P_4 + 3P_5 + 2P_6 + P_7\right) & M = 8 \\ \frac{1}{2}\left(\sum_{k=1}^{8} P_k + \sum_{k=2}^{5} P_k + P_5 + 2P_6 + P_7\right) & M = 16 \end{cases} . \tag{3.11}$$

The BEP for OC can be evaluated from (3.11) and (3.10).

3.2.3 Compact Expressions for SEP and BEP

There is a similarity between the expressions for SEP (3.6) and BEP (3.10). In fact, if one defines

$$C\left(\phi, \xi\right) = \frac{1}{\pi} \int \left[\int_0^{\phi} M_{\gamma_t|\boldsymbol{\lambda}} \left(-\frac{\xi}{\sin^2\theta} \right) d\theta \right] p_{\boldsymbol{\lambda}}\left(\boldsymbol{\lambda}\right) d\boldsymbol{\lambda}, \tag{3.12}$$

then the SEP in (3.6) can be expressed as

$$P_{\text{s},M\text{-PSK}} = C\left(\left(M-1\right)\pi/M, \sin^2\left(\pi/M\right)\right); \tag{3.13}$$

whereas the P_k in (3.10) is given by

$$\begin{aligned} P_k &= \frac{1}{2} C\left(\pi\left[1-(2k-1)/M\right], \sin^2\left[(2k-1)\pi/M\right]\right) \\ &\quad -\frac{1}{2} C\left(\pi\left[1-(2k+1)/M\right], \sin^2\left[(2k+1)\pi/M\right]\right). \end{aligned} \tag{3.14}$$

Evaluation of $C\left(\phi, \xi\right)$ is carried out next.

3.2.4 Evaluation of $C\left(\phi, \xi\right)$

Start by substituting (3.7) in (3.12),

$$\begin{aligned} C\left(\phi, \xi\right) &= \frac{1}{\pi} \int \left\{ \int_0^{\phi} \left(\frac{\sin^2\theta}{\sin^2\theta + \xi\gamma} \right)^{N_A - N_{\min}} \left[\prod_{i=1}^{N_{\min}} \left(\frac{\sin^2\theta}{\sin^2\theta + \xi\frac{P_s}{\lambda_i}} \right) \right] d\theta \right\} \\ &\quad \times p_{\boldsymbol{\lambda}}\left(\boldsymbol{\lambda}\right) d\boldsymbol{\lambda}. \end{aligned} \tag{3.15}$$

The direct evaluation of (3.15) is computationally intensive even for small $N_{\min}$ since it involves a $(N_{\min}+1)$-fold integration. It will be shown that an expression for $C(\phi,\xi)$ can be obtained which involves only a single integration form.

By converting the product in (3.15) into a summation,

$$\begin{aligned} & C(\phi,\xi) \\ = & \frac{1}{\pi}\int\left\{\int_0^{\phi}\left(\frac{\sin^2\theta}{\sin^2\theta+\xi\gamma}\right)^{N_A-N_{\min}}\left[\sum_{n=1}^{N_{\min}}A_n(\boldsymbol{\lambda})\frac{\left(\sin^2\theta\right)^{N_{\min}}}{\sin^2\theta+\xi\frac{P_s}{\lambda_n}}\right]d\theta\right\}p_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})\,d\boldsymbol{\lambda} \\ = & \sum_{n=1}^{N_{\min}}\int A_n(\boldsymbol{\lambda})\,V(\lambda_n)\,p_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})\,d\boldsymbol{\lambda}, \end{aligned} \tag{3.16}$$

where

$$A_n(\boldsymbol{\lambda}) = \frac{\lambda_n^{N_{\min}-2}\prod_{i=1}^{N_{\min}}\lambda_i}{(\xi P_s)^{N_{\min}-1}}\prod_{i=1}^{n-1}\left(\frac{1}{\lambda_n-\lambda_i}\right)\prod_{i=n+1}^{N_{\min}}\left(\frac{1}{\lambda_n-\lambda_i}\right) \tag{3.17}$$

$$V(\lambda_n) = \frac{1}{\pi}\int_0^{\phi}\left(\frac{\sin^2\theta}{\sin^2\theta+\xi\gamma}\right)^{N_A-N_{\min}}\frac{\left(\sin^2\theta\right)^{N_{\min}}}{\sin^2\theta+\xi\frac{P_s}{\lambda_n}}d\theta. \tag{3.18}$$

By starting with (3.16) and following the same procedure detailed in Appendix C, $C(\phi,\xi)$ can be expressed as

$$C(\phi,\xi)=\left(\frac{1}{\xi\beta}\right)^{N_{\min}-1}\sum_{p=0}^{N_{\min}-1}\left(\frac{\beta}{\gamma}\right)^{p}\sum_{q=0}^{N_{\min}-1}(-1)^{N_{\min}-1+q}H_{p,q}\Upsilon_q, \tag{3.19}$$

where

$$\beta=\frac{P_s}{P_I}$$

is the signal-to-interference ratio (SIR). And $H_{p,q}$ is a sequence indexed by p and q defined by (2.41) in Chapter 2, and Υ_q is a sequence defined by

$$\Upsilon_q=\int_0^{\infty}z_{N_{\min}}^{N_{\max}-N_{\min}+q}\left(z_{N_{\min}}+\frac{\sigma^2}{P_I}\right)^{N_{\min}-1}e^{-z_{N_{\min}}}V\left(P_I z_{N_{\min}}+\sigma^2\right)dz_{N_{\min}}, \tag{3.20}$$

which can be evaluated as (see Appendix F)

$$\begin{aligned}\Upsilon_q &= \sum_{k=0}^{N_{\min}} \binom{N_A}{k} (-\xi_1)^k \left[\sum_{i=1}^{N_{\min}-k} F_{N_{\min}-k-i} X_{q,i-1} + Y_{q,N_{\min}-k} \right] \\ &+ \sum_{k=N_{\min}+1}^{N_A} \binom{N_A}{k} (-\xi_1)^k \left[- \sum_{i=0}^{k-N_{\min}-1} G_{k-N_{\min}-i} X_{q,-(i+1)} + Y_{q,N_{\min}-k} \right],\end{aligned} \tag{3.21}$$

where $\xi_1 = \xi\gamma$. Other terms (F, X, Y and G) in (3.21) are defined as (m is an integer)

$$\begin{aligned}F_m &= \frac{1}{\pi} \sum_{l=0}^{m} \binom{m}{l} \xi_1^{m-l} \\ &\times \left[\frac{1}{2^{2l}} \binom{2l}{l} \phi + \frac{(-1)^l}{2^{2l-1}} \sum_{k=0}^{l-1} (-1)^k \binom{2l}{k} \frac{\sin(2l-2k)\,\phi}{2l-2k} \right]\end{aligned} \tag{3.22}$$

$$X_{q,m} = \xi_1^m \sum_{l=0}^{N_{\min}-1-m} \binom{N_{\min}-1-m}{l} \left(\frac{\beta}{\gamma}\right)^l (N_{\max}+q-l-1)! \tag{3.23}$$

$$\begin{aligned}Y_{q,m} &= \frac{1}{\pi} \xi_1^m \sqrt{\frac{1}{\xi\beta}} \int_0^\infty \frac{\left(z_{N_{\min}} + \frac{\beta}{\gamma}\right)^{N_{\min}-m}}{\sqrt{z_{N_{\min}} + \xi\beta + \frac{\beta}{\gamma}}} \operatorname{arctg}\left(\sqrt{\frac{1}{\xi\beta}\left(z_{N_{\min}} + \xi\beta + \frac{\beta}{\gamma}\right)} \operatorname{tg}\phi \right) \\ &\times \exp\left(-z_{N_{\min}}\right) \left(z_{N_{\min}}\right)^{N_{\max}-N_{\min}+q+m} dz_{N_{\min}}\end{aligned} \tag{3.24}$$

$$G_m = \frac{1}{\pi}\frac{1}{\xi_1^m}\frac{1}{\left(1+\xi_1^{-1}\right)^{m-\frac{1}{2}}}\sum_{l=0}^{m-1}\binom{m-1}{l}\binom{2l}{l}\left(\frac{\xi_1^{-1}}{4}\right)^l \times\left[\mathrm{tg}^{-1}\left(\sqrt{1+\xi_1^{-1}}\mathrm{tg}\phi\right)+\sqrt{1+\xi_1^{-1}}\frac{\mathrm{tg}\phi}{2}\right. \times\left.\sum_{j=1}^{l}\frac{4^j}{\binom{2j}{j}\,j\left(1+\left(1+\xi_1^{-1}\right)\mathrm{tg}^2\phi\right)^j}\right]. \tag{3.25}$$

By applying the transformation $z_{N_{\min}} = \mathrm{tg}\varphi$ to (3.24),

$$Y_{q,m} = \frac{1}{\pi}\xi_1^m\sqrt{\frac{1}{\xi\beta}}\int_0^{\frac{\pi}{2}}\frac{\left(\mathrm{tg}\varphi+\frac{\beta}{\gamma}\right)^{N_{\min}-m}}{\sqrt{\mathrm{tg}\varphi+\xi\beta+\frac{\beta}{\gamma}}}\mathrm{arctg}\left(\sqrt{\frac{1}{\xi\beta}\left(\mathrm{tg}\varphi+\xi\beta+\frac{\beta}{\gamma}\right)}\mathrm{tg}\phi\right) \times\exp\left(-\mathrm{tg}\varphi\right)\left(\mathrm{tg}\varphi\right)^{N_{\max}-N_{\min}+q+m}\sec^2\varphi d\varphi. \tag{3.26}$$

The finite integral above is readily evaluated numerically.

An inspection of the terms that make up $C(\phi,\xi)$ in (3.19) indicates that the integration in (3.26) is the only one required to evaluate $C(\phi,\xi)$. With (3.13) and (3.19), the SEP can be calculated. BEP can be calculated with (3.11), (3.14) and (3.19). Although (3.19) and the related expressions appear involved, they consist of elementary functions and a single integral form which can be readily computed numerically using Matlab or similar software.

These expressions are exact. But since the calculation of $Y_{q,m}$ in (3.26) involves integration, the actual accuracy of the final result will depend on the accuracy of the numerical integration.

3.3 Asymptotic Expressions of BEP for High SNR

In this section, closed-form expressions of BEP for asymptotically high SNR γ are derived. The asymptotic expressions provide an intuitive insight and are easy to calculate. These expressions are needed later to compare the performance of OC with that of MSDD. The case of no interference (MRC) is discussed first.

3.3.1 Asymptotic Expressions for No Interference and SNR $\gg 1$

In this case $N_{\min} = 0$. From (3.7) it follows that

$$M_{\gamma_t}(s) = \left(\frac{1}{1-\gamma s}\right)^{N_A}. \tag{3.27}$$

And (3.6) becomes

$$\begin{aligned} P_{\text{s},M\text{-PSK}} &= \frac{1}{\pi}\int_0^{(M-1)\pi/M} M_{\gamma_t|\boldsymbol{\lambda}}\left(-\frac{\sin^2(\pi/M)}{\sin^2\theta}\right) d\theta \\ &= \frac{1}{\pi}\int_0^{(M-1)\pi/M} \left(\frac{1}{1+\gamma\frac{\sin^2(\pi/M)}{\sin^2\theta}}\right)^{N_A} d\theta. \end{aligned} \tag{3.28}$$

For Gray coding and high SNR, there is only one bit error for each symbol error, hence the relation between SEP $P_{\text{s},M\text{-PSK}}$ and BEP $P_{\text{b},M\text{-PSK}}$ is

$$\begin{aligned} P_{\text{b},M\text{-PSK}} &\approx \frac{1}{\log_2 M} P_{\text{s},M\text{-PSK}} \\ &= \frac{1}{\log_2 M}\frac{1}{\pi}\int_0^{(M-1)\pi/M} \left(\frac{1}{1+\gamma\frac{\sin^2(\pi/M)}{\sin^2\theta}}\right)^{N_A} d\theta. \end{aligned} \tag{3.29}$$

For $\gamma \gg 1$,

$$\begin{aligned} P_{\text{b},M\text{-PSK}} &\approx \frac{1}{\log_2 M}\frac{1}{\pi}\int_0^{(M-1)\pi/M} \left(\frac{1}{\gamma\frac{\sin^2(\pi/M)}{\sin^2\theta}}\right)^{N_A} d\theta \\ &= \frac{1}{\log_2 M}\frac{1}{\left[\gamma\sin^2(\pi/M)\right]^{N_A}}\frac{1}{\pi}\int_0^{(M-1)\pi/M} \sin^{2N_A}\theta d\theta. \end{aligned} \tag{3.30}$$

The integration in the above equation is treated separately for $M = 2$ and $M \geqslant 4$.

- **BPSK,** $M = 2$

By using Eq. (2.513) in [31],

$$\frac{1}{\pi}\int_0^{\pi/2} \sin^{2N_A}\theta d\theta = \frac{1}{2^{2N_A+1}}\binom{2N_A}{N_A}. \tag{3.31}$$

Since

$$\begin{aligned}\binom{2N_A}{N_A} &= \frac{(2N_A)!}{N_A!N_A!} \\ &= \frac{2N_A}{N_A}\frac{(2N_A-1)!}{(N_A-1)!N_A!} \\ &= 2\binom{2N_A-1}{N_A-1},\end{aligned} \tag{3.32}$$

$$\frac{1}{\pi}\int_0^{\pi/2} \sin^{2N_A}\theta d\theta = \frac{1}{4^{N_A}}\binom{2N_A-1}{N_A-1}. \tag{3.33}$$

Substitute (3.33) into (3.30), then the BEP for BPSK (which is equal to SEP) is obtained as

$$P_{\text{b,BPSK}} \approx \binom{2N_A-1}{N_A-1}\frac{1}{(4\gamma)^{N_A}}. \tag{3.34}$$

- M**-PSK,** $M \geqslant 4$

When $M = 4$ (QPSK), define a function $\rho(N_A)$ such that

$$\frac{1}{\pi}\int_0^{3\pi/4} \sin^{2N_A}\theta d\theta = \rho(N_A)\frac{1}{2^{2N_A}}\binom{2N_A-1}{N_A-1}. \tag{3.35}$$

The relation between $\rho(N_A)$ and N_A is shown in Figure 3.1 and the following table.

Table 3.1 and Figure 3.1 show that $\rho(N_A) \approx 2$ for $N_A \geq 2$. Therefore (3.35) becomes

$$\frac{1}{\pi}\int_0^{3\pi/4} \sin^{2N_A}\theta d\theta \approx 2\frac{1}{2^{2N_A}}\binom{2N_A-1}{N_A-1}. \tag{3.36}$$

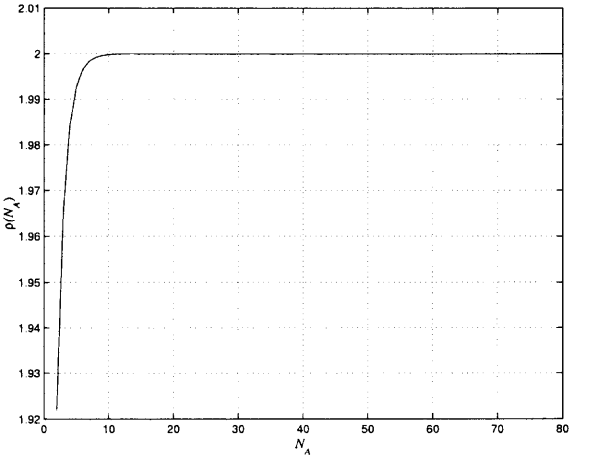

Figure 3.1 $\rho(N_A)$ versus N_A.

Table 3.1 The relation between $\rho(N_A)$ and N_A

N_A	2	3	4	5	6	10	20	80
$\rho(N_A)$	1.92	1.97	1.99	1.99	2.00	2.00	2.00	2.00

When $M \geqslant 4$, since $\frac{3\pi}{4} \leqslant \frac{(M-1)\pi}{M} < \pi$ and $\sin^{2N_A}\theta > 0$,

$$\frac{1}{\pi}\int_0^{3\pi/4} \sin^{2N_A}\theta d\theta \leqslant \frac{1}{\pi}\int_0^{(M-1)\pi/M} \sin^{2N_A}\theta d\theta < \frac{1}{\pi}\int_0^{\pi} \sin^{2N_A}\theta d\theta. \tag{3.37}$$

Since

$$\begin{aligned}\frac{1}{\pi}\int_0^{\pi} \sin^{2N_A}\theta d\theta &= \frac{1}{\pi}\int_0^{\pi/2} \sin^{2N_A}\theta d\theta + \frac{1}{\pi}\int_{\pi/2}^{\pi} \sin^{2N_A}\theta d\theta \\ &= \frac{1}{\pi}\int_0^{\pi/2} \sin^{2N_A}\theta d\theta + \frac{1}{\pi}\int_{-\pi/2}^{0} \sin^{2N_A}(\theta+\pi)\, d(\pi+\theta) \\ &= 2\frac{1}{\pi}\int_0^{\pi/2} \sin^{2N_A}\theta d\theta, \end{aligned} \tag{3.38}$$

by using (3.33),

$$\frac{1}{\pi}\int_0^{\pi} \sin^{2N_A}\theta d\theta = 2\frac{1}{2^{2N_A}}\binom{2N_A-1}{N_A-1}. \tag{3.39}$$

It follows that in (3.37), $\frac{1}{\pi}\int_0^{(M-1)\pi/M} \sin^{2N_A}\theta d\theta$, for $M \geqslant 4$, is "sandwiched" between two quantities approximately equal to $2\frac{1}{2^{2N_A}}\binom{2N_A-1}{N_A-1}$, hence

$$\frac{1}{\pi}\int_0^{(M-1)\pi/M} \sin^{2N_A}\theta d\theta \approx 2\frac{1}{2^{2N_A}}\binom{2N_A-1}{N_A-1}. \tag{3.40}$$

Substitute (3.40) into (3.30), then the BEP for $M \geqslant 4$ is

$$\begin{aligned} P_{\text{b},M\text{-PSK}} &\approx \frac{1}{\log_2 M}\frac{1}{\left[\gamma \sin^2(\pi/M)\right]^{N_A}} 2\frac{1}{2^{2N_A}}\binom{2N_A-1}{N_A-1} \\ &= \frac{2}{\log_2 M}\binom{2N_A-1}{N_A-1}\frac{1}{\left[4\gamma \sin^2(\pi/M)\right]^{N_A}}. \end{aligned} \tag{3.41}$$

In the numerical section, it is proven that for high SNR $\gamma \gg 1$, the approximate BEP expressions in (3.34) and (3.41) are very close to the BEP yielded by (3.29).

3.3.2 Asymptotic Expressions for $N_A > N_I$, SIR$\ll 1$, and SNR $\gg 1$

In this case, since SIR$\ll 1, \lambda_i \gg P_s$, from (3.7) it follows that

$$M_{\gamma_t|\boldsymbol{\lambda}}(s) \approx \left(\frac{1}{1-\gamma s}\right)^{N_A-N_I}. \tag{3.42}$$

The difference between the MGF for SIR $\ll 1$ in (3.42) and the MGF without interference in (3.27) is the loss of $N_{\min}$ diversity degrees of freedom in (3.42). Therefore the BEP for SIR $\ll 1$ can be obtained by replacing N_A in (3.34) and (3.41) with $N_A - N_I$, i.e., for OC in the presence of N_I interference source, when SNR $\gg 1$ and SIR $\ll 1$,

$$P_{\text{b,BPSK}} \approx \binom{2(N_A - N_I) - 1}{(N_A - N_I) - 1} \frac{1}{(4\gamma)^{N_A - N_I}} \quad \text{for BPSK} \tag{3.43}$$

and

$$P_{\text{b},M\text{-PSK}} \approx \frac{2}{\log_2 M} \binom{2(N_A - N_I) - 1}{(N_A - N_I) - 1} \frac{1}{\left[4\gamma \sin^2(\pi/M)\right]^{N_A - N_I}} \quad \text{for } M\text{-PSK}. \tag{3.44}$$

3.4 Numerical Results

In this section, numerical results are used to demonstrate the new exact SEP and BEP expressions. The interference generated in the simulations had a Gaussian distribution as assumed in the analysis. Analytical results were calculated using (3.13) (for SEP) and (3.11) (for BEP) and related expressions such as (3.19) and (3.14). To facilitate the comparison, both simulation results and analysis results are presented in all figures.

Figure 3.2 shows the SEP versus symbol SNR $= P_s/\sigma^2$ for $N_A = 6$ branches, $N_I = 4$ interferers and SIR $= P_s/P_I = 10$ dB. Figures 3.3 and 3.4 show BEP versus bit SNR $= P_s/\sigma^2/\log_2(M)$. Figure 3.3 is for $N_A = 6$ branches, $N_I = 4$ interferers and SIR $= 0$ dB. Figure 3.4 is for $N_A = 4$ branches, $N_I = 6$ interferers and SIR $= 15$ dB. In Figure 3.4, since there are insufficient degrees of freedom to completely suppress the interference, the BEP reaches an error floor as SNR increases.

Figure 3.5 shows the BEP versus SIR for $N_A = 4$ branches, $N_I = 6$ interferers and bit SNR $= 10$ dB. Figure 3.6 shows the BEP versus the number of receive branches N_A for $N_I = 4$ interferers, bit SNR $= 10$ dB and SIR $= 15$ dB. It can be seen that $\log_{10}$(BEP) decreases linearly with the increase in branches.

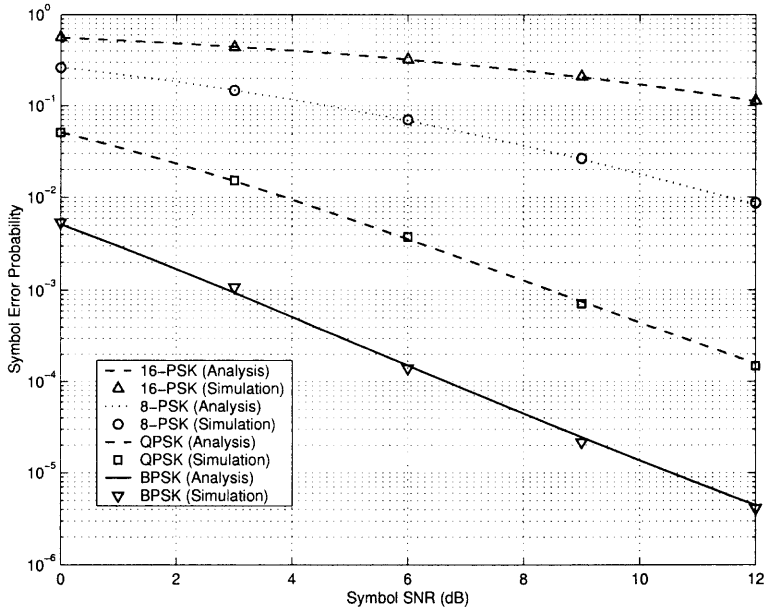

Figure 3.2 SEP versus symbol SNR for $N_A = 6$ branches, $N_I = 4$ interferers, SIR = 10 dB.

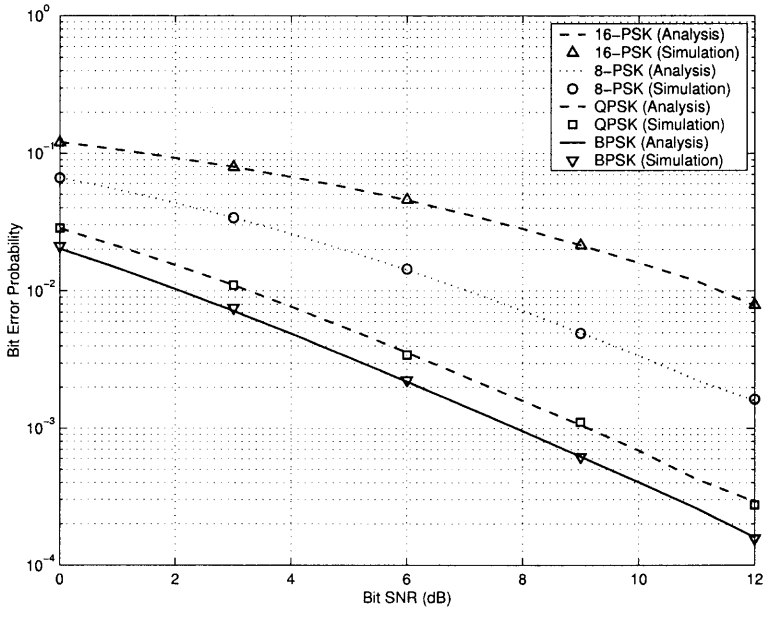

Figure 3.3 BEP versus bit SNR for $N_A = 6$ branches, $N_I = 4$ interferers, SIR = 0 dB.

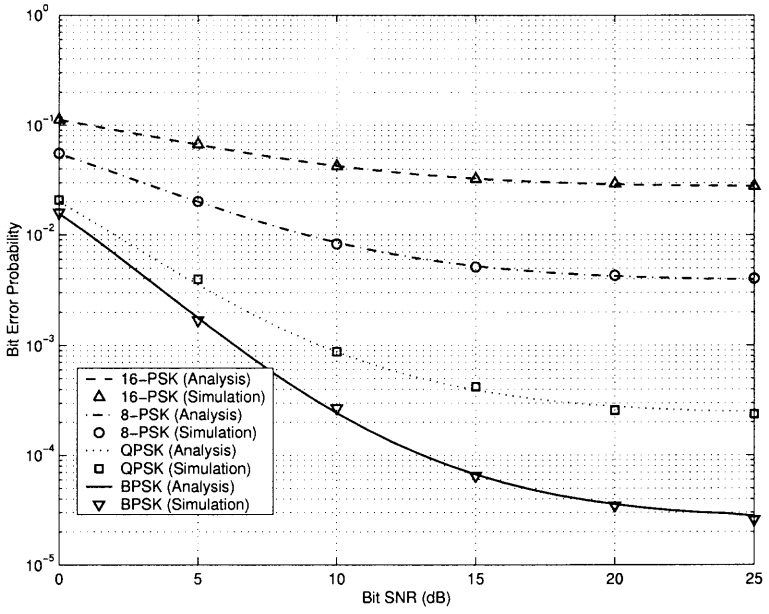

Figure 3.4 BEP versus bit SNR for $N_A = 4$ branches, $N_I = 6$ interferers, SIR = 15 dB.

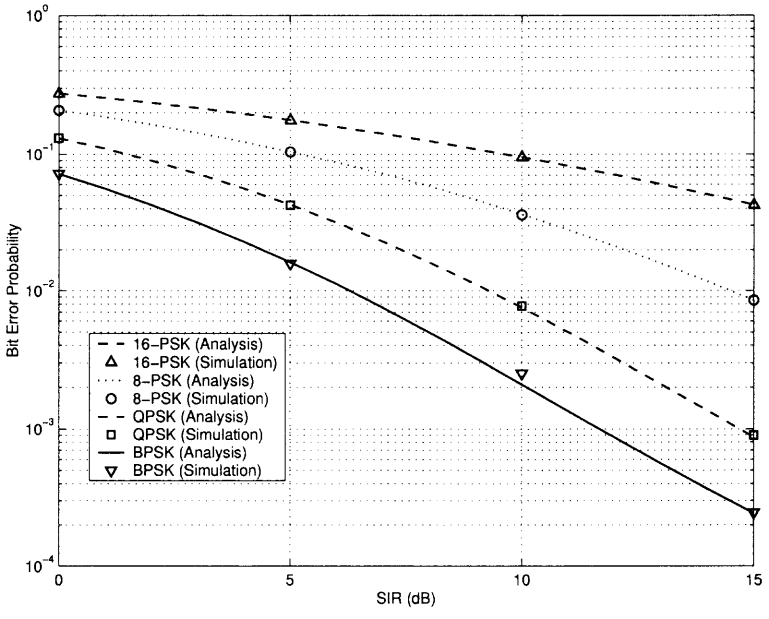

Figure 3.5 BEP versus SIR for $N_A = 4$ branches, $N_I = 6$ interferers, bit SNR = 10 dB.

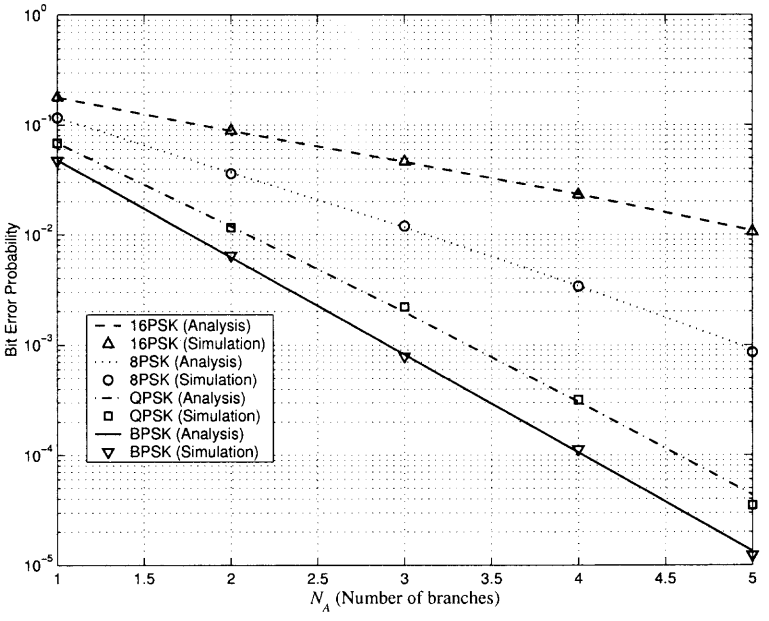

Figure 3.6 BEP versus the number of branches N_A, $N_I = 4$ interferers, bit SNR = 10 dB, SIR = 15 dB.

In all figures, the analysis results match the simulation results. This provides a convincing demonstration of the validity of the analytical expressions developed in this chapter.

Figure 3.7 shows that when there is no interference, the asymptotic results yielded by (3.34) (for BPSK) and (3.41) (for 8-PSK) are very close to the non-asymptotic results yielded by (3.29) for SNR > 15 dB.

Figures 3.8 and 3.9 show the results for the case with interference. The asymptotic results are yielded by (3.43) (for BPSK) and (3.44) (for 8-PSK). The exact results are yielded by (3.11) and related expressions such as (3.19) and (3.14). For SIR = 0 dB and BPSK in Figure 3.8, the asymptotic results are not very close to the exact results since SIR is not much less than 1. For all of the other cases shown, the asymptotic results are very close to exact results for SNR > 15 dB.

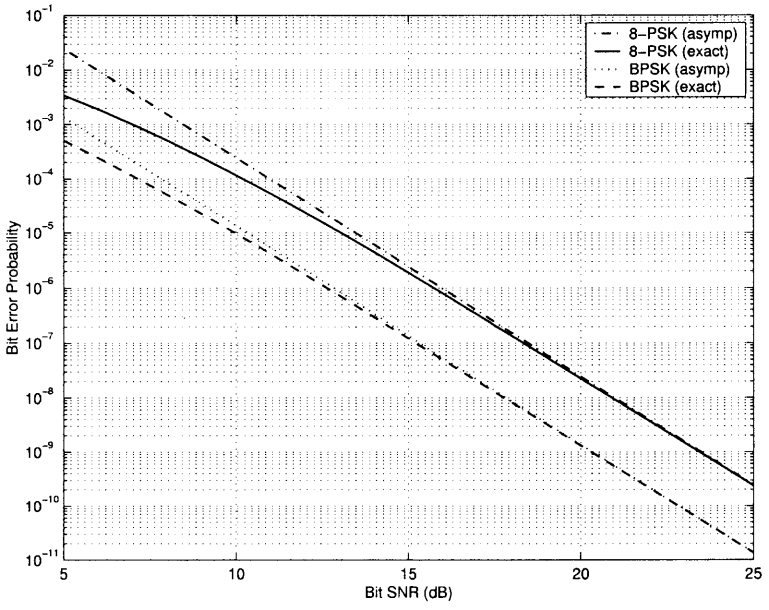

Figure 3.7 Comparison of asymptotic results and exact results, no interference, $N_A = 4$ branches.

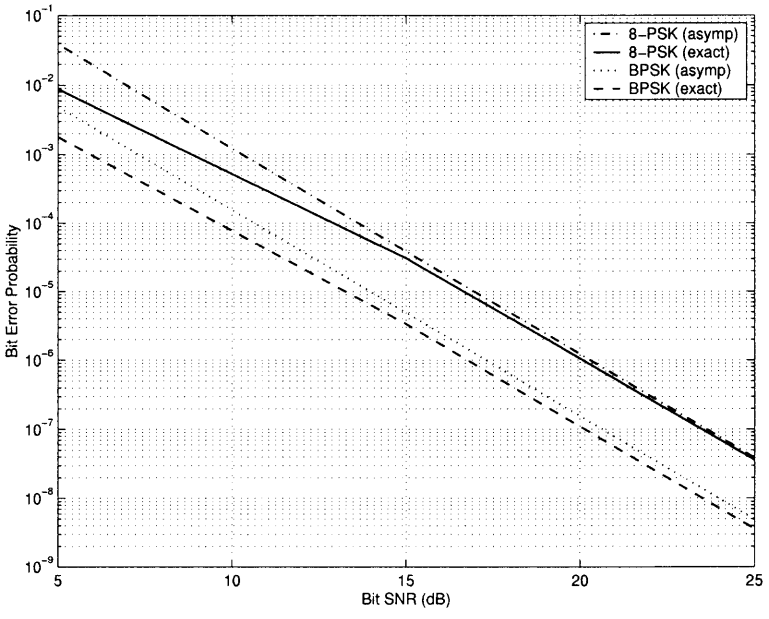

Figure 3.8 Comparison of asymptotic results and exact results, $N_A = 4$ branches, $N_I = 2$ interferers, SIR $= -0$ dB.

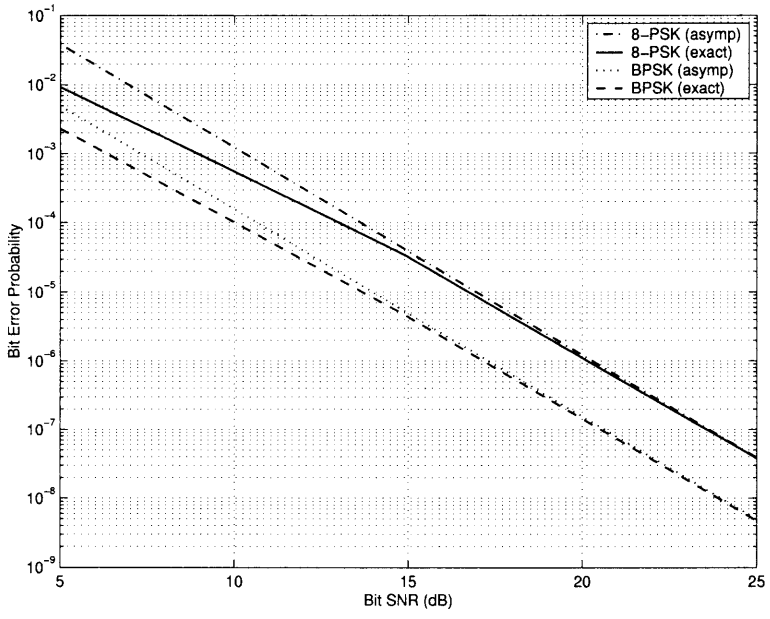

Figure 3.9 Comparison of asymptotic results and exact results, $N_A = 4$ branches, $N_I = 2$ interferers, SIR $= -10$ dB.

CHAPTER 4

SEP OF OC WITH ARBITRARY INTERFERENCE POWER

4.1 Introduction

In Chapters 2, 3 and almost all other literature that focuses on the discussion of BEP and SEP for OC, the power levels of the interferers are often assumed to be equal. In this chapter, expressions for the case of arbitrary interference power levels are derived.

In most of the literature, with some exception such as [32] where the SEP was derived from the decision metric of OC, the MGF approach is exploited to derive SEP for systems with multiple interferers. The MGF approach is very popular for evaluating the SEP since it requires the MGF of the SNR or the SINR instead of the respective PDF. In many cases, the MGF has a much simpler form than the PDF.

It turns out that for OC, a simple expression for the PDF of SINR can be obtained from the reliability function, which is defined as the probability that the SINR is less than a threshold [33]. Subsequently, the average SEP can be obtained by using the conventional method, i.e., by averaging the instantaneous SEP over the PDF of SINR. In [34] this approach was adapted to analyze the error probability, but the author used an approximate relation between the SEP and SINR, and therefore obtained only approximate expressions.

In this chapter, the exact relation between the instantaneous SEP and the SINR is used to derive an expression for the exact SEP for M-PSK modulation. The final expression involves only a single integral with finite limits and a finite integrand. A closed-form expression for the SEP of BPSK modulation is also derived.

The system model and assumption for this chapter are the same as those described in Chapter 2, except that: (1) the desired signal s is an M-PSK symbol;

(2) the N_I interferers may have unequal power levels. The sampled output of the N_A receive branches is

$$\mathbf{r} = \sqrt{P_s}\mathbf{c}s + \sum_{i=1}^{N_I} \sqrt{P_i}\mathbf{c}_i s_i + \mathbf{n}, \tag{4.1}$$

where P_i is the i-th interferer's power; the definition of other parameters and the assumptions can be found in Chapter 2.

Define SNR

$$\gamma = \frac{P_s}{\sigma^2} \tag{4.2}$$

and the signal to interference ratio (SIR) for the i-th interferer as

$$\beta_i = \frac{P_s}{P_i}. \tag{4.3}$$

The reliability function $R(\gamma_t)$ is defined as the probability that the output SINR of OC is greater than a threshold γ_t [33]:

$$\begin{aligned} R(\gamma_t) &= \exp\left(-\frac{\gamma_t}{\gamma}\right) \sum_{m=1}^{N_A-N_I} \frac{1}{(m-1)!} \left(\frac{\gamma_t}{\gamma}\right)^{m-1} \\ &\quad + \exp\left(-\frac{\gamma_t}{\gamma}\right) \sum_{m=\max(N_A-N_I+1,1)}^{N_A} \sum_{i=0}^{N_A-m} \frac{C_i}{(m-1)!} \frac{\gamma_t^i}{\prod_{n=1}^{N_I}(\gamma_t+\beta_n)} \left(\frac{\gamma_t}{\gamma}\right)^{m-1}, \end{aligned} \tag{4.4}$$

where C_i is the coefficient of γ_t^i in power series expression of the product $\prod_{n=1}^{N_I}(\gamma_t+\beta_n)$, i.e., $\prod_{n=1}^{N_I}(\gamma_t+\beta_n) = \sum_{i=0}^{N_I} C_i\gamma_t^i$, from which it can be proven by the method of mathematical induction that

$$C_i = \sum_{\substack{n_1+n_2+\cdots+n_{N_I}=N_I-i \\ n_i\in\{0,1\}}} \beta_1^{n_1}\beta_2^{n_2}\cdots\beta_{N_I}^{n_{N_I}}. \tag{4.5}$$

The PDF of the output SINR γ_t is

$$p_{\gamma_t}(\gamma_t) = -\frac{dR(\gamma_t)}{d\gamma_t}. \tag{4.6}$$

The SEP expressions for M-PSK and BPSK are developed in Sections II and III, respectively. Numerical results are shown in Section IV.

4.2 SEP Analysis for M-PSK Modulation

For OC with M-PSK modulation, the SEP conditioned on the output SINR γ_t can be written as [24]

$$P_{\mathrm{s},M\text{-PSK}}\left(E|\gamma_t\right) = \frac{1}{\pi}\int_0^{(M-1)\pi/M} \exp\left\{-\gamma_t \frac{\sin^2\left(\pi/M\right)}{\sin^2\theta}\right\} d\theta, \tag{4.7}$$

where M is the number of symbols in M-PSK modulation. In order to get the ensemble average SEP $P_{\mathrm{s},M\text{-PSK}}$, $P_{\mathrm{s},M\text{-PSK}}\left(E|\gamma_t\right)$ needs to be averaged over the PDF $p_{\gamma_t}\left(\gamma_t\right)$ of γ_t,

$$P_{\mathrm{s},M\text{-PSK}} = \int_0^{\infty} P_{\mathrm{s},M\text{-PSK}}\left(E|\gamma_t\right) p_{\gamma_t}\left(\gamma_t\right) d\gamma_t. \tag{4.8}$$

Substitute (4.6) in (4.8) and use the method of integration by parts. Then

$$\begin{aligned} P_{\mathrm{s},M\text{-PSK}} &= \frac{M-1}{M} \\ &\quad + \int_0^{\infty} R\left(\gamma_t\right) \frac{1}{\pi}\int_0^{\frac{(M-1)\pi}{M}} \exp\left(-\gamma_t \frac{\sin^2\left(\pi/M\right)}{\sin^2\theta}\right)\left(-\frac{\sin^2\left(\pi/M\right)}{\sin^2\theta}\right) d\theta d\gamma_t \end{aligned} \tag{4.9}$$

Define $x = \mathrm{ctg}\theta$, the above equation becomes

$$\begin{aligned} P_{\mathrm{s},M\text{-PSK}} &= \frac{M-1}{M} - \int_0^{\infty} R\left(\gamma_t\right) \sin^2\left(\pi/M\right) \exp\left(-\sin^2\left(\pi/M\right)\gamma_t\right) \\ &\quad \times \frac{1}{\pi}\int_{\mathrm{ctg}\frac{(M-1)\pi}{M}}^{\infty} \exp\left[-\sin^2\left(\pi/M\right)\gamma_t x^2\right] dx d\gamma_t \\ &= \frac{M-1}{M} - \int_0^{\infty} R\left(\gamma_t\right) \sqrt{\frac{\sin^2\left(\pi/M\right)}{\pi\gamma_t}} e^{-c\gamma_t} \\ &\quad \times Q\left[\sqrt{2\sin^2\left(\pi/M\right)\gamma_t}\,\mathrm{ctg}\frac{\left(M-1\right)\pi}{M}\right] d\gamma_t. \end{aligned} \tag{4.10}$$

Let $\gamma_t = \tan^2\varphi$, then the final expression for the average SEP can be obtained as

$$P_{\text{s},M\text{-PSK}} = \frac{M-1}{M} - 2\sqrt{\frac{\sin^2(\pi/M)}{\pi}}\int_0^{\frac{\pi}{2}} R\left(\tan^2\varphi\right)\exp\left[-\sin^2(\pi/M)\tan^2\varphi\right] \times Q\left[\sqrt{2\sin^2(\pi/M)}\tan\varphi\text{ctg}\frac{(M-1)\pi}{M}\right]\left(1+\tan^2\varphi\right)d\varphi. \quad (4.11)$$

The above expression only involves integration of a finite integrand over finite limits; therefore its evaluation is fast and accurate. It can be used for any M, including $M = 2$, i.e., BPSK modulation.

4.3 SEP Analysis for BPSK Modulation

Though (4.11) is easy enough to evaluate, it would be more desirable to obtain a closed-form expression, since closed-form expressions are usually faster when it comes to numerical evaluation. Up until now, (4.11) is the best that can be obtained for M-PSK when $M \neq 2$; whereas for BPSK, a closed-form expression can be derived.

In the case of BPSK, $M = 2$, $Q\left[\sqrt{2\sin^2(\pi/M)\,\gamma_t}\tan\varphi\text{ctg}\frac{(M-1)\pi}{M}\right] = Q(0) = \frac{1}{2}$, (4.10) becomes

$$P_{\text{s,BPSK}} = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{1}{\pi}}\int_0^{\infty} R(\gamma_t)\,\gamma_t^{-\frac{1}{2}}e^{-\gamma_t}d\gamma_t. \quad (4.12)$$

Substitute (4.4) in (4.12). Then

$$P_{\text{s,BPSK}} = \frac{1}{2} - \Upsilon_1 - \Upsilon_2, \quad (4.13)$$

where

$$\Upsilon_1 = \frac{1}{2}\sqrt{\frac{1}{\pi}}\sum_{m=1}^{N_A-N_I}\frac{1}{(m-1)!}\frac{1}{\gamma^{m-1}}\int_0^{\infty}\gamma_t^{m-\frac{3}{2}}e^{-\alpha\gamma_t}d\gamma_t \quad (4.14)$$

$$\Upsilon_2 = \frac{1}{2}\sqrt{\frac{1}{\pi}}\sum_{m=\max(N_A-N_I+1,1)}^{N_A}\sum_{i=0}^{N_A-m}\frac{C_i}{(m-1)!}\frac{1}{\gamma^{m-1}} \times\int_0^{\infty}\frac{\gamma_t^{m+i-\frac{3}{2}}}{\prod_{n=1}^{N_I}(\gamma_t+\beta_n)}e^{-\alpha\gamma_t}d\gamma_t \quad (4.15)$$

$$\alpha = 1+\frac{1}{\gamma}. \quad (4.16)$$

By using Eq. 3.381.4 and Eq. 8.339 in [31], Υ_1 can easily be expressed in the following closed form:

$$\begin{aligned}\Upsilon_1 &= \frac{1}{2}\sqrt{\frac{\gamma}{\gamma+1}}\sum_{m=0}^{N_A-N_I-1}\frac{(2m-1)!!}{2^m m!}\frac{1}{(\gamma+1)^m} \\ &= \frac{1}{2}\sqrt{\frac{\gamma}{\gamma+1}}\sum_{m=0}^{N_A-N_I-1}\binom{2m}{m}\left[\frac{1}{4(\gamma+1)}\right]^m. \end{aligned} \tag{4.17}$$

Next Υ_2 will be expressed in closed form.

4.3.1 Expansion of $1/\prod_{n=1}^{N_I}(\gamma_t+\beta_n)$

The evaluation of Υ_2 involves the integration over the rational function $1/\prod_{n=1}^{N_I}(\gamma_t+\beta_n)$ that may have more than a single pole. To facilitate the integration, this function needs to be expanded into the summation of rational functions, each with a single pole.

The N_I SIR terms β_n $(n=1,2,\cdots,N_I)$ may or may not be equal to each other. They may be separated into N_p sets, with the SIR's in the same set equal to each other. The SIR's from different sets are unequal. More specifically, assume the first set contains n_1 SIR β_n's with each equaling β_1'; the second set contains n_2 SIR β_n's with each equaling $\beta_2';\cdots$; the N_p-th set contains n_{N_p} β_n's with each equaling β_{N_p}'. Then $n_1+n_2+\cdots+n_{N_p}=N_I$ and $\beta_1',\beta_2',\cdots,\beta_{N_p}'$ are distinct. With this assumption

$$\begin{aligned}\frac{1}{\prod_{n=1}^{N_I}(\gamma_t+\beta_n)} &= \frac{1}{\prod_{k=1}^{N_p}(\gamma_t+\beta_k')^{n_k}} \\ &= \sum_{k=1}^{N_p}\sum_{l=1}^{n_k}\frac{A_{k,l}}{(\gamma_t+\beta_k')^l}, \end{aligned} \tag{4.18}$$

where

$$A_{k,l} = \frac{1}{(n_k-l)!}\frac{d^{(n_k-l)}}{d\gamma_t^{(n_k-l)}}\frac{(\gamma_t+\beta_k')^{n_k}}{\prod_{i=1}^{N_p}(\gamma_t+\beta_k')^{n_i}}\bigg|_{\gamma_t=-\beta_k'}. \tag{4.19}$$

Two special cases are used to illustrate this procedure.

- All the SIR's are equal

In this case, there is only one unique β. Therefore $N_p = 1, n_1 = N_I, \beta_1' = \beta_1$. Hence

$$\frac{1}{\prod_{n=1}^{N_I}(\gamma_t + \beta_n)} = \frac{1}{(\gamma_t + \beta_1)^{N_I}}, \tag{4.20}$$

and

$$A_{k,l} = \begin{cases} 1 & k = 1, l = N_I \\ 0 & \text{others} \end{cases}. \tag{4.21}$$

- All the SIR's are unique

In this case $\beta_i \neq \beta_j$ for any $i \neq j$. Hence $\beta_n' = \beta_n$ for $n = 1, 2, \cdots, N_I$, $N_p = N_I, n_k = 1$ for $k = 1, 2, \cdots, N_p$, and

$$\frac{1}{\prod_{n=1}^{N_I}(\gamma_t + \beta_n)} = \sum_{k=1}^{N_I} \frac{A_{k,1}}{\gamma_t + \beta_k}, \tag{4.22}$$

where

$$\begin{aligned} A_{k,1} &= \left. \frac{(\gamma_t + \beta_k)}{\prod_{i=1}^{N_I}(\gamma_t + \beta_i)} \right|_{\gamma_t = -\beta_k} \\ &= \frac{1}{\prod_{i=1, i\neq k}^{N_I}(\beta_i - \beta_k)}. \end{aligned} \tag{4.23}$$

4.3.2 Evaluation of the Integration in Equation (4.15)

Substitute (4.18) into (4.15),

$$\Upsilon_2 = \frac{1}{2}\sqrt{\frac{1}{\pi}} \sum_{m=\max(N_A - N_I + 1, 1)}^{N_A} \sum_{i=0}^{N_A - m} \frac{C_i}{(m-1)!} \frac{1}{\gamma^{m-1}} \sum_{k=1}^{N_p} \sum_{l=1}^{n_k} A_{k,l} B_{m,i,k,l}, \tag{4.24}$$

where

$$\begin{aligned} B_{m,i,k,l} &= \int_0^\infty \frac{\gamma_t^{m+i-\frac{3}{2}}}{\left(\gamma_t+\beta_k'\right)^l} e^{-\alpha\gamma_t} d\gamma_t \\ &= \int_0^\infty \left[\frac{\gamma_t^{m+i-1}}{\left(\gamma_t+\beta_k'\right)^l}\right] \gamma_t^{-\frac{1}{2}} e^{-\alpha\gamma_t} d\gamma_t \\ &= \int_0^\infty \left[\sum_{p=\max(l-(m+i-1),1)}^{l} \frac{D_{m,i,k,l,p}}{\left(\gamma_t+\beta_k'\right)^p} + \sum_{q=0}^{m+i-1-l} F_{m,i,k,l,q}\gamma_t^q\right] \\ &\quad \times \gamma_t^{-\frac{1}{2}} e^{-\alpha\gamma_t} d\gamma_t, \end{aligned} \tag{4.25}$$

in which $D_{m,i,k,l,p}$ and $F_{m,i,k,l,q}$ are evaluated as:

$$\begin{aligned} D_{m,i,k,l,p} &= \frac{1}{(l-p)!}\frac{d^{(l-p)}}{d\gamma_t^{(l-p)}}\left\{\left(\gamma_t+\beta_k'\right)^l\left[\frac{\gamma_t^{m+i-1}}{\left(\gamma_t+\beta_k'\right)^l} - \sum_{q=0}^{m+i-1-l} F_{m,i,k,l,q}\gamma_t^q\right]\right\}\Bigg|_{\gamma_t=-\beta_k'} \\ &= \frac{1}{(l-p)!}\frac{d^{(l-p)}}{d\gamma_t^{(l-p)}}\gamma_t^{m+i-1}\Bigg|_{\gamma_t=-\beta_k'} \\ &= \frac{1}{(l-p)!}\frac{(m+i-1)!}{[m+i-1-(l-p)]!}\left(-\beta_k'\right)^{m+i-1-(l-p)} \end{aligned} \tag{4.26}$$

$$\begin{aligned} F_{m,i,k,l,q} &= -\frac{1}{q!}\frac{d^q}{d\gamma_t^q}\left[\frac{\gamma_t^{m+i-1}}{\left(\gamma_t+\beta_k'\right)^l} - \sum_{p=\max(l-(m+i-1),1)}^{l}\frac{D_{m,i,k,l,p}}{\left(\gamma_t+\beta_k'\right)^p}\right]\Bigg|_{\gamma_t=0} \\ &= -\frac{1}{q!}\sum_{p=\max(l-(m+i-1),1)}^{l}(-1)^{m+i-1-l+p+q}\left(\beta_k'\right)^{m+i-1-l-q} \\ &\quad \times \frac{1}{(l-p)!}\frac{(m+i-1)!}{[m+i-1-(l-p)]!}\frac{(p+q-1)!}{(p-1)!}. \end{aligned} \tag{4.27}$$

The $B_{m,i,k,l}$ in (4.25) can be further expressed as

$$B_{m,i,k,l} = \sum_{p=\max(l-(m+i-1),1)}^{l} D_{m,i,k,l,p} 2\left(\beta_k'\right)^{\frac{1}{2}-p} G_{k,p} + \sum_{q=0}^{m+i-1-l} F_{m,i,k,l,q} H_q, \tag{4.28}$$

where

$$G_{k,p} = \frac{1}{2\left(\beta_k'\right)^{\frac{1}{2}-p}} \int_0^\infty \frac{1}{\left(\gamma_t + \beta_k'\right)^p} \gamma_t^{-\frac{1}{2}} e^{-\alpha\gamma_t} d\gamma_t \tag{4.29}$$

$$\begin{aligned} H_q &= \int_0^\infty \gamma_t^{q-\frac{1}{2}} e^{-\alpha\gamma_t} d\gamma_t \\ &= \sqrt{\frac{\pi}{\alpha}} \frac{(2q)!}{(4\alpha)^q q!}. \end{aligned} \tag{4.30}$$

In deriving (4.30), Eq. 3.381.4 and Eq. 8.339 in [31] are used.

Let $\gamma_t = \beta_k' y^2$, then (4.29) becomes

$$G_{k,p} = \int_0^\infty \frac{1}{(y^2+1)^p} e^{-\alpha\beta_k' y^2} dy, \tag{4.31}$$

from which

$$G_{k,0} = \frac{1}{2}\sqrt{\frac{\pi}{\alpha\beta_k'}} \tag{4.32}$$

$$G_{k,1} = \frac{\pi}{2} e^{\alpha\beta_k'} \operatorname{erfc}\left(\sqrt{\alpha\beta_k'}\right) \tag{4.33}$$

and

$$G_{k,p} = \left[1 - \frac{1}{2(p-1)} - \frac{\alpha\beta_k'}{(p-1)}\right] G_{k,p-1} + \frac{\alpha\beta_k'}{(p-1)} G_{k,p-2} \tag{4.34}$$

for $p \geq 2$. The above equation shows that the value of $G_{k,p}$ for $p \geq 2$ can be evaluated recursively from the values of $G_{k,p-1}$ and $G_{k,p-2}$ (e.g., $G_{k,2} = \left(\frac{1}{2} - \alpha\beta_k'\right) G_{k,1} + \alpha\beta_k' G_{k,0}$). Since (4.34) is a second order difference equation with initial values $G_{k,0}$ and $G_{k,1}$, $G_{k,p}$ can be solved by the method detailed in [27]. The solution is omitted here since it is not as simple as (4.34) for evaluating $G_{k,p}$.

To summarize, by combining together (4.13) and the related expressions (4.17), (4.24), (4.28), and (4.30), the closed-form expressions for the SEP of BPSK can be

obtained as

$$
\begin{aligned}
P_{\text{s,BPSK}} &= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\gamma}{\gamma+1}} \sum_{m=0}^{N_A-N_I-1} \binom{2m}{m} \left[\frac{1}{4(\gamma+1)}\right]^m \\
&\quad - \frac{1}{2}\sqrt{\frac{1}{\pi}} \sum_{m=\max(N_A-N_I+1,1)}^{N_A} \sum_{i=0}^{N_A-m} \frac{C_i}{(m-1)!} \frac{1}{\gamma^{m-1}} \\
&\quad \times \sum_{k=1}^{N_p} \sum_{l=1}^{n_k} A_{k,l} \left[\sum_{p=\max(l-(m+i-1),1)}^{l} 2\left(\beta_k'\right)^{\frac{1}{2}-p} D_{m,i,k,l,p} G_{k,p} \right. \\
&\quad \left. + \sum_{q=0}^{m+i-1-l} F_{m,i,k,l,q} \sqrt{\frac{\pi}{\alpha}} \frac{(2q)!}{(4\alpha)^q q!} \right]
\end{aligned} \tag{4.35}
$$

with C_i defined in Section 4.1, $A_{k,l}$ defined in (4.19), $D_{m,i,k,l,p}$ defined in (4.26) and $G_{k,p}$ given in (4.32) to (4.34).

4.3.3 Special Case: No Interferer

In this case, $N_I = 0$, (4.35) becomes

$$
P_{\text{s,BPSK}} = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\gamma}{\gamma+1}} \sum_{m=0}^{N_A-1} \binom{2m}{m} \left[\frac{1}{4(\gamma+1)}\right]^m, \tag{4.36}
$$

which is the same as Eq. (C-18) in [2].

4.3.4 Special Case: One Interferer

In this case, $N_I = 1, N_p = 1, n_1 = 1, A_{k,l} = 1, \beta_k' = \beta$ (the SIR of the only interferer), (4.35) becomes

$$\begin{aligned} P_{\text{s,BPSK}} &= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\gamma}{\gamma+1}} \sum_{m=0}^{N_A-2} \binom{2m}{m} \left[\frac{1}{4(\gamma+1)}\right]^m \\ &\quad - \frac{1}{2}\sqrt{\frac{1}{\pi}} \frac{C_0}{(N_A-1)!} \frac{1}{\gamma^{N_A-1}} \\ &\quad \times \left[2\beta^{\frac{1}{2}-1} D_{N_A,0,1,1,1} G_{1,1} + \sum_{q=0}^{N_A-2} F_{N_A,0,1,1,q} \sqrt{\frac{\pi}{\alpha}} \frac{(2q)!}{(4\alpha)^q q!}\right] \\ &= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\gamma}{\gamma+1}} \sum_{m=0}^{N_A-2} \binom{2m}{m} \left[\frac{1}{4(\gamma+1)}\right]^m \\ &\quad - \frac{1}{2} \frac{1}{(N_A-1)!} \left(-\frac{\beta}{\gamma}\right)^{N_A-1} \\ &\quad \times \left[\sqrt{\pi\beta} e^{\alpha\beta} \text{erfc}\left(\sqrt{\alpha\beta}\right) - \sqrt{\frac{1}{\alpha}} \sum_{q=0}^{N_A-2} \frac{(2q)!}{q!} \left(-\frac{1}{4\alpha\beta}\right)^q\right], \end{aligned} \tag{4.37}$$

which is the same as Eq. (16) in [11].

The SEP of BPSK can be calculated from both (4.13) and (4.35). The evaluation of (4.35) takes less time for small N_A (number of receive branches) and N_I (number of interferers). For large N_A and N_I, (4.35) is less accurate than (4.13) due to its complexity and the numerical accuracy of the Matlab software being used.

4.4 Numerical Results

In this section, both analysis results and simulation results are provided. The analysis results were calculated using (4.11), and are represented by continuous curves while Monte Carlo simulation results are indicated by discrete symbols.

Figure 4.1 shows the SEP versus SNR for QPSK modulation, 4 branches, varying number of interferers, and SIR = 10 dB for each interferer. Figure 4.2 shows the SEP versus SNR for $N_A = 8$ branches and $N_I = 6$ interferers. The SIR's for the 6

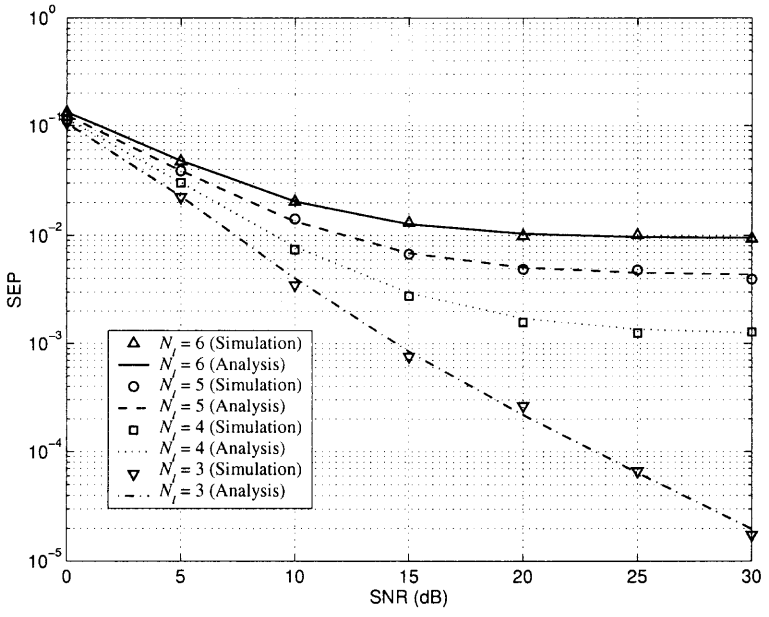

Figure 4.1 SEP versus SNR for QPSK modulation, $N_A = 4$ branches and SIR = 10 dB for each interferer. The number of interferers varies from $N_I = 3$ to $N_I = 6$.

interferers are 10 dB, 10 dB, 2 dB, 2 dB, 0 dB, 0 dB, respectively. Figure 4.3 shows the SEP versus SIR for BPSK modulation, $N_I = 9$ interferers and SNR = 10 dB, with the number of branches varying from 7 to 10. The SIR's of 7 interferers are fixed to 0 dB, while the SIR's for the other 2 interferers vary from 0 dB to 30 dB as shown by the x-axis. Figure 4.4 shows the SEP versus the number of branches N_A for 32 interferers at fixed SNR = 10 dB. The SIR's for 16 interferers are 0 dB, while the SIR's for the other 16 interferers are 2 dB.

The analysis results match the simulation results in all figures that cover various configurations. This proves that the analytical expression can be used to evaluate the SEP.

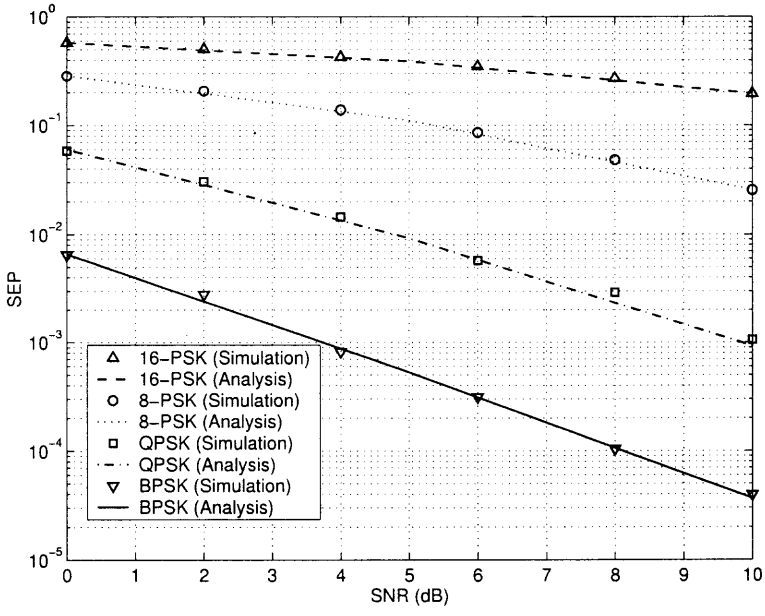

Figure 4.2 SEP versus SNR for $N_A = 8$ branches and $N_I = 6$ interferers. The SIR's for the 6 interferers are 10 dB, 10 dB, 2 dB, 2 dB, 0 dB, 0 dB, respectively.

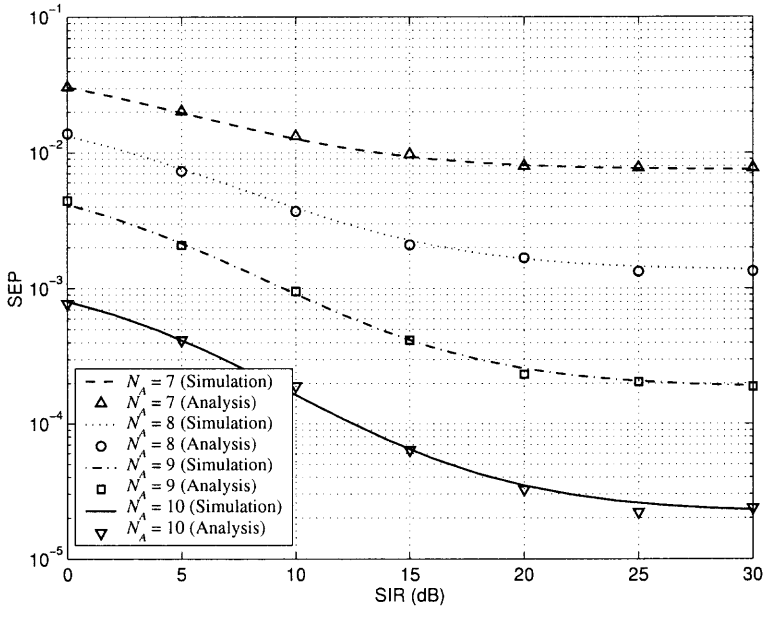

Figure 4.3 SEP versus SIR for BPSK modulation, $N_I = 9$ interferers and SNR = 10 dB. The number of branches varies from $N_A = 7$ to $N_A = 10$. The SIR's for 7 interferers are fixed to 0 dB. The SIR's for the other 2 interferers vary from 0 dB to 30 dB as shown by the abscissa.

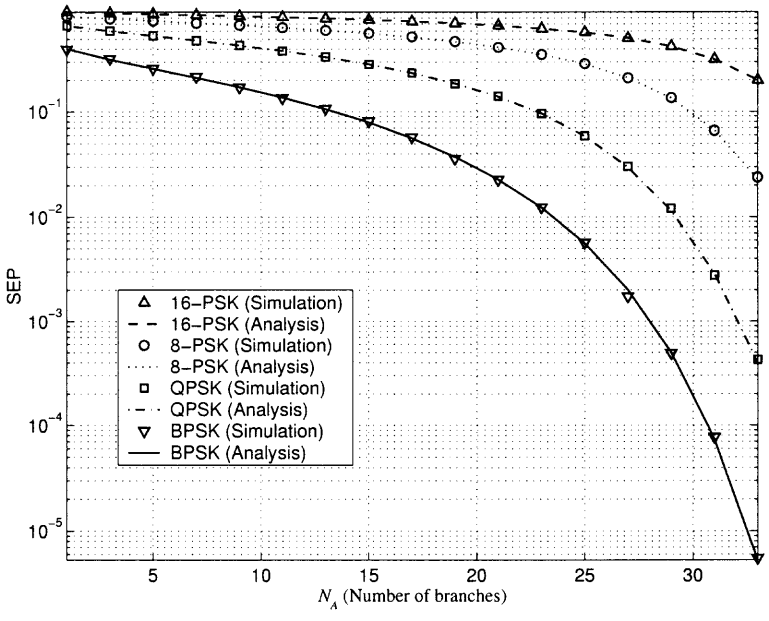

Figure 4.4 SEP versus the number of branches N_A for $N_I = 32$ interferers and SNR = 10 dB. The SIR's for 16 interferers are 0 dB, while the SIR's for the other 16 interferers are 2 dB.

CHAPTER 5

MSDD WITH KNOWN COVARIANCE MATRIX OF INTERFERENCE PLUS NOISE

5.1 Introduction

In the previous chapters, the performance of OC was analyzed. OC is a coherent detection technique which requires information on the channel phase of the desired signal. In this and the next chapter, the non-coherent detection scheme multiple symbol differential detection (MSDD) is presented.

Multiple symbol differential detection (MSDD) was first proposed for detecting M-PSK signals transmitted over an additive white Gaussian noise (AWGN) channel [35]. The main advantage of MSDD is that it does not require a coherent phase reference at the receiver. It does require, however, the ability to measure relative phase differences. MSDD performs maximum-likelihood detection of a block of information symbols based on a corresponding observation interval. The method was presented to bridge the gap between the performance of coherent detection of M-PSK and conventional differential detection of M-DPSK [35]. The channel phase was assumed to be unknown to the receiver, but constant over multiple symbol intervals. In [35] it was shown that for a long observation interval, the performance of MSDD (in terms of the required SNR for a given BEP) approached that of coherent detection (with differential encoding at the transmitter).

The MSDD scheme was extended to trellis coded M-PSK in [36]. MSDD for the fading channel was analyzed in [37] and for correlated fading in [38]. MSDD application to multiuser code division multiple access (CDMA) was considered in [39]. Performance of MSDD with narrow-band interference over a non-fading channel

was discussed in [40]. A system with MSDD and receive diversity was formulated in [41], while [42, 43] considered MSDD with transmit diversity.

In this chapter, an extension to MSDD is derived for communication in the presence of interferers. The channel of the desired signal is a diversity Rayleigh channel with multiple outputs. For an antenna array at the receiver of a communication system operating over a slow fading channel, any signal source is spatially correlated. The channel realizations at each output are mutually independent, constant over the observation interval, and unknown to the receiver. The Gaussian assumption is made with respect to the aggregate of interference plus noise. The covariance matrix of the interference plus noise is assumed to be known. The MSDD decision statistic is derived based on the principle of MLSD. A closed-form expression for the conditional pairwise error probability (PEP) is derived. A closed-form expression for the BEP is intractable; however, one is obtained for an approximation to the union bound. The approximation utilizes only dominant terms in the union bound and is shown to be a good approximation of the BEP for some cases.

The computational complexity of direct computation of the decision statistic grows exponentially with the number of symbols in the observation interval. For single channel MSDD, an optimum algorithm was proposed in [44]. Sub-optimal decision feedback algorithms for the single channel case were suggested in [45, 46, 47]. No efficient MSDD algorithm was published for MSDD with diversity. In this chapter, the sub-optimal decision feedback algorithm in [47] is applied to MSDD with diversity. The main improvement over published algorithms is the introduction of iterations for symbol detection.

This chapter is organized as follows: Section 5.2 presents the signal model. The MSDD decision statistic is derived in Section 5.3. The error analysis is developed in Section 5.4, while Section 5.5 presents the numerical results.

5.2 System Model

Since there are some differences and some additional assumptions for MSDD, the system model that was initially described in Section 1.2 needs to be expanded. For a wireless communications system with N_A independent receive branches, the sampled output of the matched filter corresponding to time k and the l-th branch is

$$r_{k,l} = \sqrt{P_s} c_l s_k + z_{k,l}, \quad l = 1, 2, \cdots, N_A, \tag{5.1}$$

where P_s is the power of the desired signal, c_l is the channel gain of the l-th branch, s_k is the transmitted M-DPSK symbol, and $z_{k,l}$ is interference plus noise. For M-DPSK modulation, the transmitted signals can be expressed as $s_k = e^{j\theta_k}$, $\theta_k = 2\pi (i_k - 1)/M$, $i_k = 1, 2, \cdots, M$. The transmitted symbols are differentially encoded, i.e., $\theta_k = \theta_{k-1} + \Delta\theta_k$, where $\Delta\theta_k$ is the phase representing the transmitted information at time k.

The signal model in vector notation is

$$\mathbf{r}_k = \sqrt{P_s} \mathbf{c} s_k + \mathbf{z}_k, \tag{5.2}$$

where $\mathbf{r}_k = [r_{k,1}, \cdots, r_{k,N_A}]^T$, $\mathbf{c} = [c_1, \cdots, c_{N_A}]^T$, $\mathbf{z}_k = [z_{k,1}, \cdots, z_{k,N_A}]^T$.

The channel gains c_l's are assumed to be independent and identically distributed (i.i.d.), zero-mean, circularly symmetric, complex Gaussian random variables (Rayleigh fading), with variance $\Omega_l/2$ per dimension. The correlated noise term $\mathbf{z}_k$ is the aggregate of an interference source and AWGN. It is assumed to be complex-valued, zero-mean, circularly symmetric, and governed by a Gaussian distribution with covariance matrix $\mathbf{R} = E\left[\mathbf{z}_k \mathbf{z}_k^H\right]$. For interference of N_I sources plus AWGN, the covariance matrix can be expressed as

$$\mathbf{R} = E\left[\mathbf{z}_k \mathbf{z}_k^H\right] = P_I \sum_{i=1}^{N_I} \mathbf{c}_i \mathbf{c}_i^H + \text{diag}\left(\sigma_1^2, \sigma_2^2, \cdots, \sigma_{N_A}^2\right), \tag{5.3}$$

where the parameters are defined in Chapter 1.

Consider a sequence of K symbols running from time $k-(K-1)$ to k. Assume the channel is static over the duration of this sequence. Using vector notation,

$$\underline{\mathbf{r}}_k = \sqrt{P_s}\mathbf{H}\mathbf{s}_k + \underline{\mathbf{z}}_k, \tag{5.4}$$

where $\underline{\mathbf{r}}_k = \left[\mathbf{r}_{k-(K-1)}, \cdots, \mathbf{r}_k\right]^T$, $\mathbf{s}_k$ and $\underline{\mathbf{z}}_k$ are vectors defined similar to $\underline{\mathbf{r}}_k$. And $\mathbf{H} = \mathbf{I}_K \otimes \mathbf{c}$ is the channel matrix for the signal of interest, where the symbol '$\otimes$' denotes the Kronecker product, and $\mathbf{I}_K$ is the identity matrix of rank K.

As in Chapter 2, define $N_{\max} = \max(N_A, N_I)$ and $N_{\min} = \min(N_A, N_I)$. Diagonalize $\mathbf{R}$ as $\mathbf{R} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^H$, where $\mathbf{\Lambda} = \text{diag}\ (\lambda_1, \lambda_2, \cdots, \lambda_{N_A})$. The eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{N_{\min}}$ are random variables, while $\lambda_m = \sigma^2$ for $m = N_{\min}+1, N_{\min}+2, \cdots, N_A$. Denote the set of non-trivial eigenvalues as $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \cdots, \lambda_{N_{\min}}]^T$. The inverse covariance matrix of $\mathbf{R}$ is $\mathbf{R}^{-1} = \mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{U}^H$.

5.3 Decision Statistic

The decision statistic for a symbol sequence $\mathbf{s}_k = \left[s_{k-(K-1)}, \cdots, s_k\right]^T$ based on an observation interval of length K as embodied in the vector $\underline{\mathbf{r}}_k$, will be formulated in this section. Assume the covariance matrix of the interference plus noise $\mathbf{R}$ is known. Then the maximum-likelihood detector for the sequence $\mathbf{s}_k$ is given by

$$\widehat{\mathbf{s}}_k = \arg \max_{\mathbf{s}_k} p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{R}), \tag{5.5}$$

where $p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{R})$ is the likelihood of the observed data $\underline{\mathbf{r}}_k$ given the transmitted symbol sequence $\mathbf{s}_k$ and the covariance matrix $\mathbf{R}$. Under the Gaussian assumption for the aggregate of interference and noise, the observation $\underline{\mathbf{r}}_k$ (conditioned on the transmitted sequence $\mathbf{s}_k$, the covariance matrix $\mathbf{R}$ and channel $\mathbf{c}$) has a multivariate Gaussian distribution. The conditional probability $p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{R}, \mathbf{c})$ can then be expressed

as

$$\begin{aligned} & p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{R}, \mathbf{c}) \\ = \; & \pi^{-KN_A}|\mathbf{R}|^{-K} \\ & \times \exp\left\{-\sum_{i=0}^{K-1}(\mathbf{r}_{k-i} - \sqrt{P_s}\mathbf{c}s_{k-i})^H\mathbf{R}^{-1}(\mathbf{r}_{k-i} - \sqrt{P_s}\mathbf{c}s_{k-i})\right\}. \end{aligned} \tag{5.6}$$

Since $\mathbf{R}^{-1} = \mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{U}^H$,

$$\begin{aligned} & p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{R}, \mathbf{g}) \\ = \; & \pi^{-KN_A}|\mathbf{R}|^{-K} \\ & \times \exp\left\{-\sum_{i=0}^{K-1}(\mathbf{x}_{k-i} - \sqrt{P_s}\mathbf{g}s_{k-i})^H\mathbf{\Lambda}^{-1}(\mathbf{x}_{k-i} - \sqrt{P_s}\mathbf{g}s_{k-i})\right\}, \end{aligned} \tag{5.7}$$

where

$$\mathbf{x}_{k-i} = \mathbf{U}^H\mathbf{r}_{k-i} \tag{5.8}$$

$$\mathbf{g} = \mathbf{U}^H\mathbf{c}. \tag{5.9}$$

$\mathbf{x}_{k-i}$ is the whitened received signal vector, and $\mathbf{g}$ is the modified channel vector. Note that since $\mathbf{U}$ is unitary, the modified channel vector $\mathbf{g}$ has the same distribution as the original channel vector $\mathbf{c}$. Let the components of the modified channel vector $\mathbf{g}$ be expressed as $g_l = \alpha_l e^{j\phi_l}$, $l = 1, \ldots, N_A$. Likewise, let the l-th component of $\mathbf{x}_{k-i}$ be $x_{k-i,l}$. Expand the exponent in (5.7) and group terms that do not depend on $\mathbf{g}$ or s_{k-i},

$$\begin{aligned} p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{R}, \mathbf{g}) = \; & \pi^{-KN_A}|\mathbf{R}|^{-K}\exp\{-C_0\} \\ & \times \prod_{l=1}^{N_A}\exp\left\{-KP_s\lambda_l^{-1}\alpha_l^2 + 2\sqrt{P_s}\lambda_l^{-1}\left|y_l\left(\mathbf{s}_k\right)\right|\alpha_l\cos\left(\phi_l - \theta_l\left(\mathbf{s}_k\right)\right)\right\}, \end{aligned} \tag{5.10}$$

where

$$C_0 = \sum_{i=0}^{K-1} \sum_{l=1}^{N_A} \lambda_l^{-1} \left| x_{k-i,l} \right|^2 \tag{5.11}$$

and

$$\begin{aligned} y_l(\mathbf{s}_k) &= \left(\sum_{i=0}^{K-1} x_{k-i,l} s_{k-i}^* \right) \\ &= \left| y_l(\mathbf{s}_k) \right| e^{j\theta_l(\mathbf{s}_k)}. \end{aligned} \tag{5.12}$$

Note that $y_l(\mathbf{s}_k)$ is a function of both the transmitted sequence $\mathbf{s}_k$ and the observed sequence $\underline{\mathbf{r}}_k$.

Recall that the components of the modified channel vector $\mathbf{g}$ have the same distribution as the components of the channel vector $\mathbf{c}$. It follows that α_l is Rayleigh with $E[\alpha_l^2] = \Omega_l$ and ϕ_l is uniformly distributed in the interval $[0, 2\pi)$. To average the conditional distribution $p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{R}, \mathbf{g})$ over the modified channel $\mathbf{g}$, the integral

$$p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{R}) = \int p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{R}, \mathbf{g}) p_{\mathbf{g}}(\mathbf{g})\, d\mathbf{g} \tag{5.13}$$

needed to be evaluated, where $p_{\mathbf{g}}(\mathbf{g})$ is the PDF of $\mathbf{g}$. Assume the channels are mutually independent, then

$$p_{\mathbf{g}}(\mathbf{g}) = \prod_{l=1}^{N_A} p_{\alpha_l}(\alpha_l) p_{\phi_l}(\phi_l), \tag{5.14}$$

where $p_{\alpha_l}(\alpha_l)$ and $p_{\phi_l}(\phi_l)$ are the PDFs of α_l and ϕ_l respectively. For Rayleigh fading channels,

$$p_{\alpha_l}(\alpha_l) = \frac{2\alpha_l}{\Omega_l} \exp\left(-\frac{\alpha_l^2}{\Omega_l} \right) \quad 0 \le \alpha_l < \infty \tag{5.15}$$

$$p_{\phi_l}(\phi_l) = \frac{1}{2\pi} \qquad 0 \le \phi_l < 2\pi. \tag{5.16}$$

Substitute (5.10), (5.14), (5.15) and (5.16) into (5.13), and separate the integrations,

$$\begin{aligned} & p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{R}) \\ = \; & \pi^{-KM}|\mathbf{R}|^{-K} \exp\{-C_0\} \prod_{l=1}^{N_A} \left\{ \frac{2}{\Omega_l} \int_0^{\infty} \exp\left\{ -\left(KP_s\lambda_l^{-1} + \frac{1}{\Omega_l} \right) \alpha_l^2 \right\} \alpha_l d\alpha_l \right. \\ & \left. \times \frac{1}{2\pi} \int_0^{2\pi} \exp\left\{ 2\sqrt{P_s}\lambda_l^{-1} |y_l(\mathbf{s}_k)| \alpha_l \cos(\phi_l - \theta_l) \right\} d\phi_l \right\}. \end{aligned} \tag{5.17}$$

After the averaging over the uniform distribution of ϕ_l is carried out,

$$\begin{aligned} p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{R}) \; = \; & \pi^{-KN_A}|\mathbf{R}|^{-K} \exp\{-C_0\} \\ & \times \prod_{l=1}^{N_A} \frac{2}{\Omega_l} \int_0^{\infty} \exp\left\{ -\left(KP_s\lambda_l^{-1} + \frac{1}{\Omega_l} \right) \alpha_l^2 \right\} \\ & \times I_0(2\sqrt{P_s}\lambda_l^{-1} |y_l(\mathbf{s}_k)| \alpha_l) \alpha_l d\alpha_l, \end{aligned} \tag{5.18}$$

where $I_0(x)$ is the zeroth order modified Bessel function of the first kind. Using the integration expression of Bessel function $I_0(x)$ in Appendix G,

$$\begin{aligned} p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{R}) \; = \; & \pi^{-KN_A}|\mathbf{R}|^{-K} \exp\{-C_0\} \left(\prod_{l=1}^{N_A} \frac{\lambda_l}{KP_s\Omega_l + \lambda_l} \right) \\ & \times \exp\left\{ P_s \sum_{l=1}^{N_A} \frac{\Omega_l |y_l(\mathbf{s}_k)|^2}{\lambda_l (KP_s\Omega_l + \lambda_l)} \right\}. \end{aligned} \tag{5.19}$$

In (5.19), only the argument of the exponential function is dependent on the transmitted sequence $\mathbf{s}_k$, since only the terms $y_l(\mathbf{s}_k)$'s are functions of $\mathbf{s}_k$. Due to the monotonicity of the exponential function, maximizing $p(\underline{\mathbf{r}}_k|\mathbf{s}_k)$ with respect to $\mathbf{s}_k$ is equivalent to maximizing the following decision statistic:

$$\eta(\mathbf{s}_k) = \sum_{l=1}^{N_A} \frac{\Omega_l |y_l(\mathbf{s}_k)|^2}{\lambda_l (KP_s\Omega_l + \lambda_l)}. \tag{5.20}$$

From (5.5), the corresponding MSDD decision rule is

$$\widehat{\mathbf{s}}_k = \arg \max_{\mathbf{s}_k} \eta(\mathbf{s}_k). \tag{5.21}$$

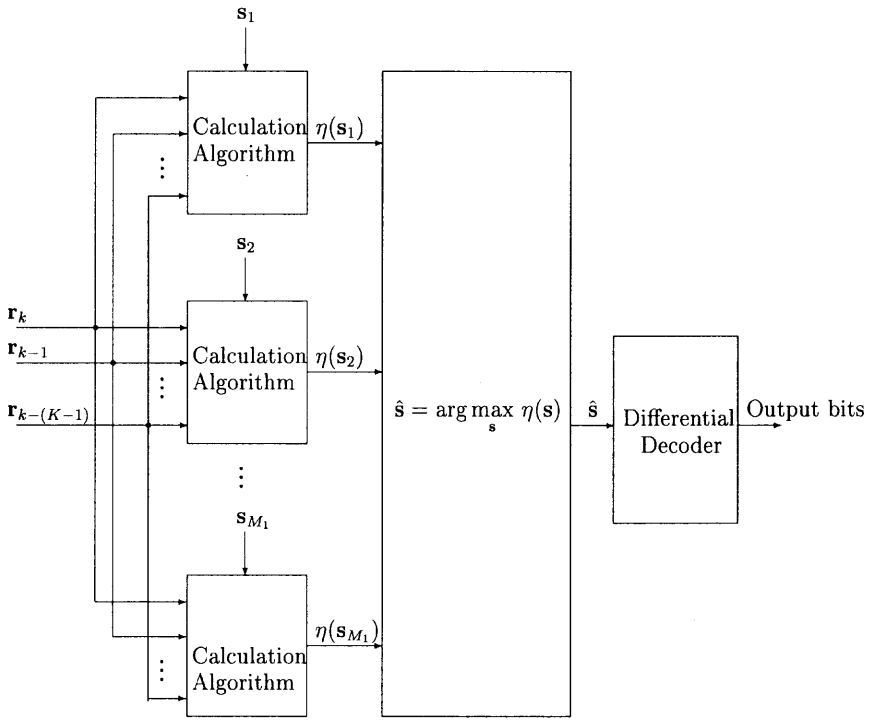

Figure 5.1 The diagram of a multiple symbol differential detector. For M-DPSK with M symbols, the number of symbol sequences that need to be tried is $M_1 = M^{K-1}$.

The detector searches through sequences $\mathbf{s}_k$ and chooses the sequence that has the largest decision metric $\eta(\mathbf{s}_k)$. A diagram of the MSDD receiver is shown in Figure 5.1.

It follows that the optimum multiple symbol differential detector for multiple channel branches in the presence of interference, is a weighted sum of correlations of whitened observations and hypothesis symbols. Note that this decision statistic does not require knowledge of the signal channel vector.

The decision statistic in (5.20) provides multiple symbol differential detection for a M-DPSK sequence transmitted over multiple independent fading channels in the presence of correlated Gaussian noise.

The decision statistic is ambiguous with respect to an arbitrary phase θ'. Indeed, let $\mathbf{s}'_k = e^{j\theta'}\mathbf{s}_k$, then

$$\begin{aligned} |y_l(\mathbf{s}'_k)| &= \left|y_l(e^{j\theta'}\mathbf{s}_k)\right| \\ &= \left|\sum_{i=0}^{K-1} x_{k-i,l}\left(e^{j\theta'}s_{k-i}\right)^*\right| \\ &= |y_l(\mathbf{s}_k)|\,. \end{aligned} \tag{5.22}$$

Differential encoding at the transmitter is required to resolve this ambiguity.

5.3.1 Iterative Decision Feedback Algorithm

The complexity of MSDD for M-DPSK with an observation interval of K symbols increases with M^{K-1}. For large K, this complexity makes simulations impractical. To overcome this difficulty, a practical sub-optimal algorithm that uses decision feedback is introduced. The basic purpose of the algorithm is to make symbol by symbol decisions rather than testing the full sequence of symbols simultaneously. The algorithm proceeds from symbol to symbol along the sequence of K symbols. At symbol i it maximizes a decision statistic assuming that the other $(K-1)$ symbols have been detected and are known. Several iterations can be carried out to improve performance. The algorithm is implemented in the following procedure:

1. Initialization:

   (a) Initialize iteration index, $m = 0$.

   (b) Initialize $\mathbf{s}_k^{(m)} \triangleq \left[1, s_{k-(K-2)}^{(m)}, s_{k-(K-3)}^{(m)}, \cdots, s_k^{(m)}\right]^T = [1, 0, 0, \cdots, 0]^T$.

   (c) Initialize time index $i = K-2$.

2. Increase iteration index $m+1 \rightarrow m$.

3. For $i = (K-2)$ to 0

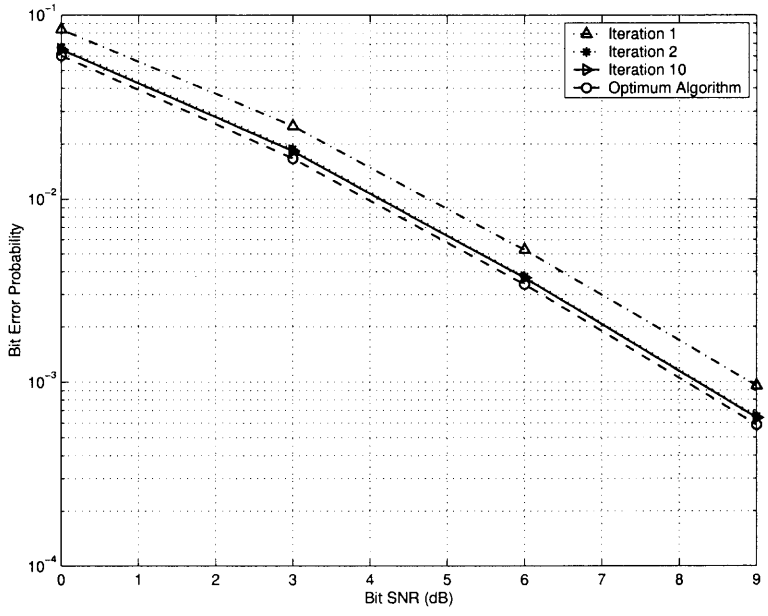

Figure 5.2 Comparison of optimum algorithm and iterative decision feedback algorithm for $L = 4$ branches, DPSK modulation, SIR $= -6$ dB.

Evaluate $s_{k-i}^{(m)} = \arg\max_{s_{k-i}^{(m)}} \eta\left(\left[1, s_{k-(K-2)}^{(m)}, \cdots, s_{k-i}^{(m)}, s_{k-i+1}^{(m-1)}, \cdots, s_{k}^{(m-1)}\right]^T\right)$.

End loop i.

4. If m is not equal to the required iteration number (which is determined empirically), go back to step 2.

5. Differentially decode $\mathbf{s}_k^{(m)}$ to get the final output.

To demonstrate the performance of this sub-optimal decision feedback algorithm, Figure 5.2 compares the performance of sub-optimal and optimal (based on (5.21)) algorithms. The comparison is for the case of $N_A = 4$ diversity branches, one interferer, binary PSK (DPSK) modulation, and SIR $= -6$ dB. For an observation interval of $K = 12$ symbols, with just 2 iterations, the performance of the sub-optimal decision feedback algorithm is within just 0.2 dB of that of the optimum algorithm. The advantage of the sub-optimal decision feedback algorithm is, of course, that it takes much less time to run than the optimum algorithm. From the figure, it can also

be observed that iterations are beneficial to the performance of decision feedback. The second iteration provides about 0.5 dB gain relative to that without iteration (iteration 1). Additional iterations do not seem to improve the performance.

5.3.2 Special Case

Some special cases provide insights into the operation of MSDD. For a channel with a flat gain profile $\Omega_l = 1$, and a flat AWGN profile $\sigma_l^2 = \sigma^2$ for $l = 1, 2, \cdots, N_A$, (5.20) can be expressed as

$$\eta(\mathbf{s}_k) = \sum_{l=1}^{N_A} \frac{|y_l(\mathbf{s}_k)|^2}{\lambda_l \left(KP_s + \lambda_l\right)}. \tag{5.23}$$

Eq. (5.23) can be further specialized in the following special cases.

- No Interference

In this case, $P_I = 0$, the noise covariance matrix is $\mathbf{R} = \sigma^2 \mathbf{I}_{N_A}$, eigenvalues $\lambda_l = \sigma^2$, $l = 1, 2, \cdots, N_A$, and $\mathbf{U} = \mathbf{I}_{N_A}$. Eq. (5.12) simplifies to

$$y_l\left(\mathbf{s}_k\right) = \sum_{i=0}^{K-1} r_{k-i,l} s_{k-i}^*. \tag{5.24}$$

The decision statistic in (5.23) becomes

$$\eta(\mathbf{s}_k) = \frac{1}{\sigma^2 \left(KP_s + \sigma^2\right)} \sum_{l=1}^{N_A} |y_l(\mathbf{s}_k)|^2. \tag{5.25}$$

Since the term outside the sum is independent of $\mathbf{s}_k$, the above decision statistic is equivalent to

$$\eta(\mathbf{s}_k) = \sum_{l=1}^{N_A} |y_l(\mathbf{s}_k)|^2 = \sum_{l=1}^{N_A} \left| \sum_{i=0}^{K-1} r_{k-i,l} s_{k-i}^* \right|^2. \tag{5.26}$$

This decision statistic is the same as that in [41, equation (8)]. Indeed (5.20) is the generalization of [41, equation (8)] to MSDD in the presence of interference.

- $N_A > N_I$ and Interference $\gg$ Noise

In this case, $N_{\min} = N_I$, $\lambda_l = \sigma^2$ for $l = N_I + 1, N_I + 2, \cdots, N_A$. It follows that the decision statistic in (5.23) becomes

$$\eta(\mathbf{s}_k) = \sum_{l=1}^{N_I} \frac{|y_l(\mathbf{s}_k)|^2}{\lambda_l \left(KP_s + \sigma^2\right)} + \sum_{l=N_I+1}^{N_A} \frac{|y_l(\mathbf{s}_k)|^2}{\sigma^2 \left(KP_s + \sigma^2\right)}. \tag{5.27}$$

For high interference to noise ratio, $\lambda_l \gg \sigma^2$ for $l = 1, 2, \cdots, N_I$. Therefore, the first summation in (5.27) can be neglected and (5.27) can be approximated by

$$\eta(\mathbf{s}_k) \approx \sum_{l=N_I+1}^{N_A} \frac{|y_l(\mathbf{s}_k)|^2}{\sigma^2 \left(KP_s + \sigma^2\right)}. \tag{5.28}$$

The interpretation of this result is that for a system with N_I strong interference sources, the decision statistic is similar to that of MSDD without interference and N_I fewer degrees of freedom. This result will be further demonstrated in the ensuing error probability analysis.

5.4 Error Probability Analysis

An exact expression for the BEP for differential detection can be obtained only for DPSK modulation and the special case of $K = 2$ symbols. The exact error analysis is intractable in the general case of MSDD with M-DPSK modulation over diversity channels and in the presence of interference. The alternative approach is to obtain an analytical approximate upper bound. An expression is derived in this section for the PEP under the Gaussian assumption for the aggregate interference plus noise. Then the union bound of the BEP can be derived from this expression. From the union bound, an approximate upper bound is derived. The approximate upper bound consists of relatively simple algebraic expressions. Even simpler expressions are obtained for large SNR and small SIR. In the numerical results section, it is shown that in many cases, the approximate upper bound is very close to the BEP obtained by simulation.

In the derivation of the PEP, assume a uniform flat power profile for the desired signal channel, $\Omega_l = 1$, and a flat AWGN profile with $\sigma_l^2 = \sigma^2$ for $l = 1, 2, \cdots, N_A$. The PEP is developed for correlated interference plus noise characterized by the covariance matrix in (5.3).

Let $\mathbf{s}_k$ and $\mathbf{s}'_k$ denote two sequences, each containing K M-DPSK symbols. Conditioned on $\boldsymbol{\lambda}$, the PEP that $\mathbf{s}_k$ is transmitted but $\mathbf{s}'_k$ is detected ($\mathbf{s}'_k \neq s_0\mathbf{s}_k$, where s_0 is an arbitrary M-PSK symbol) is denoted as $P(\mathbf{s}_k \rightarrow \mathbf{s}'_k|\boldsymbol{\lambda})$. It is conditional on the vector of random value $\boldsymbol{\lambda}$ since $\eta(\mathbf{s})$ depends on $\boldsymbol{\lambda}$.

5.4.1 Evaluation of the Conditional PEP

First evaluate the unconditional PEP $P(\mathbf{s}_k \rightarrow \mathbf{s}'_k|\boldsymbol{\lambda})$. When $\mathbf{s}_k$ is transmitted, a pairwise error event occurs when $\eta(\mathbf{s}_k) < \eta(\mathbf{s}'_k)$. Define the random variable D,

$$D = \eta(\mathbf{s}_k) - \eta(\mathbf{s}'_k). \tag{5.29}$$

Note that D is random due to both the random noise and the random channels. The probability that $D < 0$ conditioned on $\boldsymbol{\lambda}$ is evaluated now. Using steps similar to [2, Appendix B], it can be shown that

$$\begin{aligned} P(\mathbf{s}_k \rightarrow \mathbf{s}'_k|\boldsymbol{\lambda}) &= -\frac{1}{2\pi j}\int_{-\infty+j\varepsilon}^{\infty+j\varepsilon} \frac{\Phi_{D|\boldsymbol{\lambda}}(j\omega)}{\omega} d\omega \\ &= -\sum_{\mathrm{Im}(\omega_m)>0} \mathrm{Res}\left[\frac{\Phi_{D|\boldsymbol{\lambda}}(j\omega)}{\omega}; \omega_m\right], \end{aligned} \tag{5.30}$$

where ε is a small positive number; $\Phi_{D|\boldsymbol{\lambda}}(j\omega)$ is the characteristic function of D conditioned on $\boldsymbol{\lambda}$; $\mathrm{Res}\left[\frac{\Phi_{D|\boldsymbol{\lambda}}(j\omega)}{\omega}; \omega_m\right]$ denotes the residue of $\frac{\Phi_{D|\boldsymbol{\lambda}}(j\omega)}{\omega}$ at pole ω_m; and the summation is taken over the poles in the upper half of the complex plane.

In Appendix H, the following expression is derived for the characteristic function of the random variable D:

$$\Phi_{D|\boldsymbol{\lambda}}(j\omega) = \left[\frac{\omega_1\omega_2}{(\omega-\omega_1)(\omega-\omega_2)}\right]^{N_A-N_{\min}} \prod_{m=1}^{N_{\min}} \frac{\omega_{1,m}\omega_{2,m}}{(\omega-\omega_{1,m})(\omega-\omega_{2,m})}, \tag{5.31}$$

where

$$\omega_1 = -j\frac{\sqrt{\zeta^2 P_s^2 + 4\left(KP_s+\sigma^2\right)\sigma^2\zeta} - \zeta P_s}{2\zeta} \tag{5.32}$$

$$\omega_2 = j\frac{\sqrt{\zeta^2 P_s^2 + 4\left(KP_s+\sigma^2\right)\sigma^2\zeta} + \zeta P_s}{2\zeta}, \tag{5.33}$$

$$\omega_{1,m} = -j\frac{\sqrt{\zeta^2 P_s^2 + 4\left(KP_s+\lambda_m\right)\lambda_m\zeta} - \zeta P_s}{2\zeta} \tag{5.34}$$

$$\omega_{2,m} = j\frac{\sqrt{\zeta^2 P_s^2 + 4\left(KP_s+\lambda_m\right)\lambda_m\zeta} + \zeta P_s}{2\zeta} \tag{5.35}$$

for $m = 1, 2, \cdots, N_{\min}$, and

$$\zeta = K^2 - \left|v(\mathbf{s}_k, \mathbf{s}'_k)\right|^2 \tag{5.36}$$

$$v(\mathbf{s}_k, \mathbf{s}'_k) = \mathbf{s}'^H_k \mathbf{s}_k. \tag{5.37}$$

In (5.36), $v(\mathbf{s}_k, \mathbf{s}'_k)$ is the correlation coefficient between the transmitted sequence $\mathbf{s}_k$ and the detected sequence $\mathbf{s}'_k$. Note that $0 \le v(\mathbf{s}_k, \mathbf{s}'_k) < K$.

The characteristic function shown in (5.31) has the same form as the one shown in (2.9) (except for the definitions of ω_1, ω_2, etc. that are different). Therefore, the residues derived in Appendix B can be used for (5.30). From (5.30) and (B.13),

$$\begin{aligned}
&P\left(\mathbf{s}_k \to \mathbf{s}'_k|\boldsymbol{\lambda}\right) \\
&= -\sum_{m=1}^{N_{\min}} \left[\frac{\omega_1\omega_2}{\left(\omega_{2,m}-\omega_1\right)\left(\omega_{2,m}-\omega_2\right)}\right]^{N_A-N_{\min}} \frac{\omega_{1,m}}{\left(\omega_{2,m}-\omega_{1,m}\right)} \\
&\quad \times \prod_{n=1,n\neq m}^{N_{\min}} \frac{\omega_{1,n}\omega_{2,n}}{\left(\omega_{2,m}-\omega_{1,n}\right)\left(\omega_{2,m}-\omega_{2,n}\right)} \\
&\quad -\frac{\left(\omega_1\omega_2\right)^{N_A-N_{\min}}}{\left(N_A-N_{\min}-1\right)!}\left(-1\right)^{N_A-N_{\min}-1} \sum_{l=0}^{N_A-N_{\min}-1} \frac{\left(N_A-N_{\min}-1+l\right)!}{l!} \\
&\quad \times \left\{\frac{1}{\omega_2^{N_A-N_{\min}-l}} + \sum_{m=1}^{N_{\min}}\left[A_{1,m}\frac{1}{\left(\omega_2-\omega_{1,m}\right)^{N_A-N_{\min}-l}} + A_{2,m}\frac{1}{\left(\omega_2-\omega_{2,m}\right)^{N_A-N_{\min}-l}}\right]\right\} \\
&\quad \times \frac{1}{\left(\omega_2-\omega_1\right)^{N_A-N_{\min}+l}},
\end{aligned} \tag{5.38}$$

where

$$A_{1,m} = \frac{\omega_{2,m}}{(\omega_{1,m} - \omega_{2,m})} \prod_{n=1,n\neq m}^{N_{\min}} \frac{\omega_{1,n}\omega_{2,n}}{(\omega_{1,m} - \omega_{1,n})(\omega_{1,m} - \omega_{2,n})} \tag{5.39}$$

$$A_{2,m} = \frac{\omega_{1,m}}{(\omega_{2,m} - \omega_{1,m})} \prod_{n=1,n\neq m}^{N_{\min}} \frac{\omega_{1,n}\omega_{2,n}}{(\omega_{2,m} - \omega_{1,n})(\omega_{2,m} - \omega_{2,n})}. \tag{5.40}$$

5.4.2 Evaluation of the Unconditional PEP

The unconditional PEP $P(\mathbf{s}_k \rightarrow \mathbf{s}'_k)$ is obtained by averaging the conditional PEP over the random vector $\boldsymbol{\lambda}$,

$$P(\mathbf{s}_k \rightarrow \mathbf{s}'_k) = \int P(\mathbf{s}_k \rightarrow \mathbf{s}'_k|\boldsymbol{\lambda})\, p(\boldsymbol{\lambda}) d\boldsymbol{\lambda}, \tag{5.41}$$

where $p(\boldsymbol{\lambda})$ is the PDF of $\boldsymbol{\lambda}$ given in (2.21). The former expression is the exact PEP of MSDD with diversity branches and multiple interference sources. Both $P(\mathbf{s}_k \rightarrow \mathbf{s}'_k|\boldsymbol{\lambda})$ and $p(\boldsymbol{\lambda})$ are very complicated functions. Approximate expressions for the unconditional PEP are derived next.

When $N_A > N_I$, by following arguments similar to [17, 16], the following approximation can be made

$$P(\mathbf{s}_k \rightarrow \mathbf{s}'_k) \approx P(\mathbf{s}_k \rightarrow \mathbf{s}'_k|\boldsymbol{\lambda})|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}_0}, \tag{5.42}$$

where $\boldsymbol{\lambda}_0 = [\lambda_{0,1}, \lambda_{0,2}, \cdots, \lambda_{0,N_A}]^T$, and

$$\lambda_{0,l} = \begin{cases} P_I N_{\max} + \sigma^2 & l = 1, 2, \cdots, N_{\min} \\ \sigma^2 & l = N_{\min} + 1, N_I + 2, \cdots, N_A \end{cases}. \tag{5.43}$$

As mentioned in [24, p. 441] and [17], this approximation is valid for $N_I \ll N_A$ and SIR $\ll 1$. For $N_I = 1$, the approximation in (5.42) is very accurate. For other circumstances, such as $N_A < N_I$, SIR ≈ 1, it is shown in the numerical section that (5.42) is an upper bound, just as in the case of the OC detector.

The forms of both the conditional and unconditional PEP are quite complicated and do not afford much insight. It is of interest to obtain simpler expressions for special cases. In the ensuing analysis, the symbol SNR is $\gamma = P_s/\sigma^2$, and SIR is P_s/P_I.

- No Interference

  In this case $N_I = 0$. The characteristic function is

$$\Phi_D(j\omega) = \left[\frac{\omega_1\omega_2}{(\omega-\omega_1)(\omega-\omega_2)}\right]^{N_A}. \tag{5.44}$$

Note that since there is no interference, the characteristic function is unconditional. After carrying out the evaluation of the residue in (5.30),

$$P(\mathbf{s}_k \to \mathbf{s}'_k) = (-1)^{N_A-1}\sum_{l=0}^{N_A-1}\binom{N_A-1+l}{l}\frac{(\omega_1\omega_2)^{N_A}}{\omega_2^{N_A-l}(\omega_2-\omega_1)^{N_A+l}}. \tag{5.45}$$

For all the cases tried, (5.45) yielded the same numerical results as the PEP developed in [41]. However, (5.45) has the advantage of providing the PEP in closed form without the need of integration.

The case of no interference can be further simplified for large SNR $\gamma \gg 1$. For $\gamma \gg 1$, (5.32) and (5.33) can be approximated as

$$\omega_1 \approx -j\frac{K\sigma^2}{\zeta} \tag{5.46}$$

$$\omega_2 \approx j\sigma^2\gamma. \tag{5.47}$$

Substitute these results in (5.45) and notice that $\omega_2 \gg \omega_1$. Then

$$P(\mathbf{s}_k \to \mathbf{s}'_k) \approx \left(\frac{K}{\zeta\gamma}\right)^{N_A}\sum_{l=0}^{N_A-1}\binom{N_A-1+l}{l}. \tag{5.48}$$

Using Eq. (0.151) in [31],

$$P(\mathbf{s}_k \to \mathbf{s}'_k) \approx \binom{2N_A-1}{N_A}\left(\frac{K}{\zeta\gamma}\right)^{N_A}. \tag{5.49}$$

This expression clearly exhibits the N_A-order diversity of the system. It should be pointed out that since ζ is proportional to K, $P(\mathbf{s}_k \rightarrow \mathbf{s}'_k)$ does not necessarily increase with K.

- $N_A > N_I$, SIR $\ll 1$, SNR $\gg 1$

By assumption $P_I \gg P_s \gg \sigma^2$, therefore $\lambda_m \gg P_s \gg \sigma^2$. ω_1 and ω_2 can be approximated as in (5.46) and (5.47). $\omega_{1,m}$ and $\omega_{2,m}$ can be approximated as

$$\omega_{1,m} \approx -j\frac{\lambda_m}{\sqrt{\zeta}} \tag{5.50}$$

$$\omega_{2,m} \approx j\frac{\lambda_m}{\sqrt{\zeta}}. \tag{5.51}$$

Hence both $\omega_{1,m}$ and $\omega_{2,m}$ are much larger than ω_1 and ω_2. By substituting these approximate values into (5.38) and keeping only the dominant term, after some manipulations,

$$P(\mathbf{s}_k \rightarrow \mathbf{s}'_k) \approx \binom{2(N_A - N_I) - 1}{N_A - N_I} \left(\frac{K}{\zeta\gamma}\right)^{N_A - N_I}. \tag{5.52}$$

From the comparison of (5.52) and (5.49), it can be seen that the PEP for systems with diversity N_A and N_I large interference sources equals the PEP for systems with diversity $(N_A - N_I)$ without interference. This result is well known for interference suppression using OC. This analysis proves that the loss of degrees of freedom due to interference suppression carries over to MSDD.

5.4.3 BEP Approximate Upper Bound

The sequence $\mathbf{s}_k$ of M-DPSK symbols corresponds to $(K-1)\log_2 M$ information bits (with differential encoding, the first symbol is known). Let $\mathbf{u}_k$ be the sequence of $(K-1)\log_2 M$ information bits encoded as $\mathbf{s}_k$, and let $\mathbf{u}'_k$ be the sequence of information bits which results from the detection of $\mathbf{s}'_k$. The pairwise BEP associated

with transmitting a sequence $\mathbf{u}_k$ and detecting another sequence $\mathbf{u}'_k$ is given by

$$P_b\left(\mathbf{s}_k \rightarrow \mathbf{s}'_k\right) = \frac{1}{(K-1)\log_2 M} h(\mathbf{u}_k, \mathbf{u}'_k) P\left(\mathbf{s}_k \rightarrow \mathbf{s}'_k\right), \tag{5.53}$$

where $h(\mathbf{u}_k, \mathbf{u}'_k)$ denotes the Hamming distance between $\mathbf{u}_k$ and $\mathbf{u}'_k$.

The BEP that $\mathbf{s}_k$ is transmitted, and an error sequence (any error sequence) is detected, is upper bounded by the union of all pairwise bit error events. Since $\mathbf{s}_k$ can be any input sequence (e.g., the null sequence $\mathbf{s}_k = [1, 0, 0, \cdots, 0]^T$), the dependency on $\mathbf{s}_k$ can be dropped from the notation. The union bound on the BEP can then be written as

$$\begin{aligned} P_b &\leq \sum_{\mathbf{u}'_k \neq \mathbf{u}_k} P_b(\mathbf{s}_k \rightarrow \mathbf{s}'_k) \\ &= \frac{1}{(K-1)\log_2 M} \sum_{\mathbf{u}'_k \neq \mathbf{u}_k} h(\mathbf{u}_k, \mathbf{u}'_k) P\left(\mathbf{s}_k \rightarrow \mathbf{s}'_k\right), \end{aligned} \tag{5.54}$$

where the summation is taken over all the sequences $\mathbf{u}'_k$'s that are different from the transmitted sequence of information bits $\mathbf{u}_k$.

Direct application of (5.54) does not shed light on the mechanisms affecting MSDD performance. A clearer picture is obtained by developing an approximation to the union bound. Note that the union bound in (5.54) is a function of the PEP's, which in turn are determined by $\omega_1, \omega_2, \omega_{1,m}$ and $\omega_{2,m}$ (see (5.38)). From (5.32) to (5.35), $\omega_1, \omega_2, \omega_{1,m}$ and $\omega_{2,m}$ are functions of the quantity $|v(\mathbf{s}_k, \mathbf{s}'_k)|^2$ through the relation $\zeta = K^2 - |v(\mathbf{s}_k, \mathbf{s}'_k)|^2$. In [35], it is shown that on the AWGN channel, for large SNR, the dominant terms in the BEP occur for sequences for which the quantity $|v(\mathbf{s}_k, \mathbf{s}'_k)|^2$ is maximum. By carrying over the same approach to the diversity fading channel, keeping only the dominant terms, and noticing that $P\left(\mathbf{s}_k \rightarrow \mathbf{s}'_k\right)$ is constant if $|v\left(\mathbf{s}_k, \mathbf{s}'_k\right)|$ is constant, the following approximation to the union bound can be

obtained

$$A = \frac{1}{(K-1)\log_2 M}\left[\sum_{\substack{\mathbf{u}'_k \neq \mathbf{u}_k \\ |v(\mathbf{s}_k,\mathbf{s}'_k)|=|v(\mathbf{s}_k,\mathbf{s}'_k)|_{\max}}} h(\mathbf{u}_k,\mathbf{u}'_k)\right] \times \left[P\left(\mathbf{s}_k \rightarrow \mathbf{s}'_k\right)\Big|_{|v(\mathbf{s}_k,\mathbf{s}'_k)|=|v(\mathbf{s}_k,\mathbf{s}'_k)|_{\max}}\right]. \tag{5.55}$$

The maximum value of $|v(\mathbf{s}_k, \mathbf{s}'_k)|$ (for $\mathbf{s}'_k \neq s_0\mathbf{s}_k$, where s_0 is an arbitrary M-PSK symbol) was shown in [35, Eq. (38)] to be

$$|v\left(\mathbf{s}_k, \mathbf{s}'_k\right)|_{\max} = \sqrt{(K-1)^2 + 2\left(K-1\right)\left(1 - 2\sin^2\frac{\pi}{M}\right) + 1}. \tag{5.56}$$

Also, from [35, Appendix B], for sequences such that $|v\left(\mathbf{s}_k, \mathbf{s}'_k\right)| = |v\left(\mathbf{s}_k, \mathbf{s}'_k\right)|_{\max}$, the accumulated Hamming distances are

$$\sum_{\substack{\mathbf{u}'_k \neq \mathbf{u}_k \\ |v(\mathbf{s}_k,\mathbf{s}'_k)|=|v(\mathbf{s}_k,\mathbf{s}'_k)|_{\max}}} h(\mathbf{u}_k,\mathbf{u}'_k) = \begin{cases} 1, & K = 2 \\ 2\left(K-1\right), & K > 2 \end{cases} \tag{5.57}$$

for binary modulation, $M = 2$ and

$$\sum_{\substack{\mathbf{u}'_k \neq \mathbf{u}_k \\ |v(\mathbf{s}_k,\mathbf{s}'_k)|=|v(\mathbf{s}_k,\mathbf{s}'_k)|_{\max}}} h(\mathbf{u}_k,\mathbf{u}'_k) = \begin{cases} 2, & K = 2 \\ 4\left(K-1\right), & K > 2 \end{cases} \tag{5.58}$$

for multilevel modulation, $M \geqslant 4$.

Strictly speaking, (5.55) is not an upper bound of the BEP. Numerical results however, show that it is very close to, or larger than the BEP obtained by simulation. Therefore, (5.55) can be used to study the performance of MSDD in the presence of interference.

Next, the approximate upper bound for differential binary PSK (DPSK) and M–DPSK ($M \geq 4$) modulations will be evaluated.

- **DPSK** ($M = 2$)

In this case, from (5.56)

$$|v(\mathbf{s}_k, \mathbf{s}'_k)|_{\max} = K - 2. \tag{5.59}$$

For conventional differential detection, the observation interval is $K = 2$ symbols, $|v(\mathbf{s}_k, \mathbf{s}'_k)|_{\max} = 0$. In this case, there is only one error sequence; therefore the PEP is also the BEP,

$$P_b = P(\mathbf{s}_k \to \mathbf{s}'_k)|_{|v(\mathbf{s}_k, \mathbf{s}'_k)|=0}. \tag{5.60}$$

Substitute $P(\mathbf{s}_k \to \mathbf{s}'_k)$ from (5.45) into (5.60). The exact BEP for DPSK over N_A diversity fading channels without interference can be obtained. For high SNR $\gg 1$, using (5.49),

$$P_{b,\text{DPSK}} \approx \binom{2N_A - 1}{N_A} \frac{1}{(2\gamma)^{N_A}}. \tag{5.61}$$

This expression is the same as the one in [2, Eq. (14-4-28)]. It demonstrates that familiar expressions for differential detection can be obtained as a special case of the general case treated in this report.

For a longer observation interval $K > 2$, substitute (5.59) and (5.57) into (5.55) to obtain

$$A_{\text{DPSK}} = 2\, P(\mathbf{s}_k \to \mathbf{s}'_k)|_{|v(\mathbf{s}_k, \mathbf{s}'_k)|=K-2}. \tag{5.62}$$

This expression is for the approximate BEP upper bound for DPSK over slow-fading Rayleigh diversity channels with interference. Next, some special cases for $K > 2$ and SNR $\gamma \gg 1$ will be computed, resulting in simplified expressions.

No Interference, SNR $\gg 1$

Using (5.49) in (5.62), the approximate upper bound is

$$\begin{aligned} A_{\text{DPSK}} &= 2\binom{2N_A-1}{N_A}\frac{1}{(\frac{\zeta}{K}\gamma)^{N_A}}\Bigg|_{|v(\mathbf{s}_k,\mathbf{s}'_k)|=K-2} \\ &= 2\binom{2N_A-1}{N_A}\frac{1}{\left[4\left(1-\frac{1}{K}\right)\gamma\right]^{N_A}}. \end{aligned} \tag{5.63}$$

$N_A > N_I$, **SIR**$\ll 1$, **SNR** $\gg 1$

Substitute (5.52) in (5.62). The result obtained is similar to (5.63), except N_A is substituted with $(N_A - N_I)$.

$$A_{\text{DPSK}} = 2\binom{2(N_A-N_I)-1}{N_A-N_I}\frac{1}{\left[4\left(1-\frac{1}{K}\right)\gamma\right]^{N_A-N_I}}. \tag{5.64}$$

- ***M*-DPSK ($M \geqslant 4$)**

For M-DPSK, substitute (5.58) and (5.56) into (5.55). The approximate upper bound would be:

$$A_{M\text{-DPSK}} = \frac{2}{\log_2 M}\, P\left(\mathbf{s}_k \to \mathbf{s}'_k\right)\Big|_{|v(\mathbf{s}_k,\mathbf{s}'_k)|=2\left|\cos\frac{\pi}{M}\right|} \tag{5.65}$$

for $K = 2$ symbols and

$$A_{M\text{-DPSK}} = \frac{4}{\log_2 M}\, P\left(\mathbf{s}_k \to \mathbf{s}'_k\right)\Big|_{|v(\mathbf{s}_k,\mathbf{s}'_k)|=\sqrt{(K-1)^2+2(K-1)\left(1-2\sin^2\frac{\pi}{M}\right)+1}} \tag{5.66}$$

for observation intervals of size $K > 2$.

Simplified expressions for special cases are computed below.

No Interference, SNR $\gg 1$

By using (5.49), from (5.65) and (5.66), the approximate upper bound is

$$A_{M\text{-DPSK}} = \frac{2}{\log_2 M}\binom{2N_A-1}{N_A}\frac{1}{\left[2\gamma\sin^2\left(\pi/M\right)\right]^{N_A}} \tag{5.67}$$

for $K = 2$ and

$$A_{M\text{-DPSK}} = \frac{4}{\log_2 M}\binom{2N_A - 1}{N_A}\frac{1}{\left[4\left(1 - \frac{1}{K}\right)\gamma \sin^2(\pi/M)\right]^{N_A}} \tag{5.68}$$

for $K > 2$.

$N_A > N_I$, **SIR** $\ll 1$, **SNR** $\gg 1$

Substitute (5.52) in (5.65) and (5.66). A result similar to (5.67) and (5.68) is obtained, except that N_A is substituted with $(N_A - N_I)$,

$$A_{M\text{-DPSK}} = \frac{2}{\log_2 M}\binom{2(N_A - N_I) - 1}{N_A - N_I}\frac{1}{\left[2\gamma \sin^2(\pi/M)\right]^{N_A - N_I}} \tag{5.69}$$

for $K = 2$ and

$$A_{M\text{-DPSK}} = \frac{4}{\log_2 M}\binom{2(N_A - N_I) - 1}{N_A - N_I}\frac{1}{\left[4\left(1 - \frac{1}{K}\right)\gamma \sin^2(\pi/M)\right]^{N_A - N_I}} \tag{5.70}$$

for $K > 2$.

5.5 Comparison with OC

In this section, the BEP of MSDD is compared analytically with that of OC. Since the expression of the BEP for the general case is very complex, performance can be compared analytically only for the cases of small SIR (relative large interference) and no interference. Since both cases yield similar results, only the case of small SIR is treated here. Performance comparison of MSDD and OC based on numerical examples follows in the next section.

From (3.43) and (3.44) in Chapter 3, for $N_A > N_I$, SNR $\gamma \gg 1$ and SIR $\ll 1$, the BEP for OC is approximated by the expressions

$$P_{\text{b,BPSK}} \approx \binom{2(N_A - N_I) - 1}{(N_A - N_I) - 1}\frac{1}{(4\gamma)^{N_A - N_I}} \tag{5.71}$$

and

$$P_{\mathrm{b},M\text{-PSK}} \approx \frac{2}{\log_2 M}\binom{2(N_A-N_I)-1}{(N_A-N_I)-1}\frac{1}{\left[4\gamma\sin^2(\pi/M)\right]^{N_A-N_I}}. \tag{5.72}$$

Since MSDD uses M-DPSK modulation, OC and MSDD are compared on the same basis.

For OC, the exact expression of the BEP for M-DPSK is very difficult to obtain. But judging from Eq. (4.200) in [48], the BEP for M-DPSK is about twice that of M-PSK except for very small SNR. That can be demonstrated by simulation. Therefore, the BEP for M-DPSK is

$$P_{\mathrm{b},M\text{-DPSK}} \approx 2P_{\mathrm{b},M\text{-PSK}}. \tag{5.73}$$

Substitute (5.71) and (5.72) in (5.73), the BEP for OC using M-DPSK is

$$P_{\mathrm{b,DPSK}} \approx 2\binom{2(N_A-N_I)-1}{(N_A-N_I)-1}\frac{1}{(4\gamma)^{N_A-N_I}} \tag{5.74}$$

and

$$P_{\mathrm{b},M\text{-DPSK}} \approx \frac{4}{\log_2 M}\binom{2(N_A-N_I)-1}{(N_A-N_I)-1}\frac{1}{\left[4\gamma\sin^2(\pi/M)\right]^{N_A-N_I}}. \tag{5.75}$$

The ratio of the BEP of OC and the approximate upper bound of MSDD (for $K > 2$) is given by the ratios of (5.74) to (5.64) and (5.75) to (5.70), respectively

$$\frac{P_{\mathrm{b,DPSK}}}{A_{\mathrm{DPSK}}} = \frac{P_{\mathrm{b},M\text{-DPSK}}}{A_{M\text{-DPSK}}} = \left(1-\frac{1}{K}\right)^{N_A-N_I}. \tag{5.76}$$

This expression holds for $N_A > N_I$, SIR $\ll 1$ and SNR $\gg 1$. It can be concluded that for $N_A > N_I$, SIR $\ll 1$ and SNR $\gg 1$, when the observation interval of MSDD increases to infinity, i.e., $K \to \infty$, the performance of MSDD with non-coherent detection approaches that of OC with differential encoding.

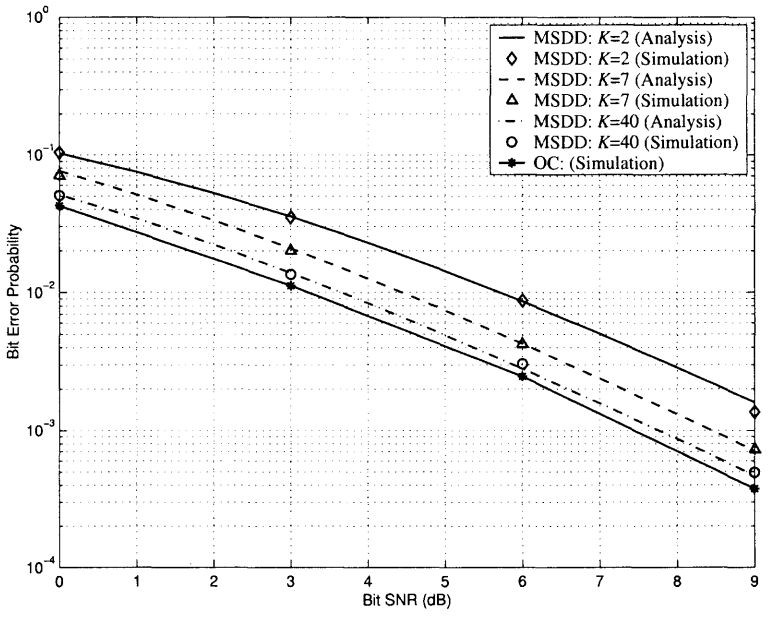

Figure 5.3 BEP versus SNR for $N_A = 4$ branches, $N_I = 1$ interferer, SIR $= -6$ dB, DPSK modulation.

According to (5.73), the BEP of OC with differential encoding is about twice that of OC without differential encoding. Therefore for MSDD of large K, the BEP is only about twice that of OC without differential encoding.

5.6 Numerical Results

Numerical results presented in this section include Monte Carlo simulation results and analysis results. In all cases, the channel branches and noise power profiles are assumed to be uniform, i.e., $\Omega_l = 1$ and $\sigma_l^2 = \sigma^2$ for $l = 1, 2, \cdots, N_A$. The bit SNR is $(P_s/\log_2(M))/\sigma^2$. For comparison purposes, BEP curves for OC with differential encoding is also provided. Except for Figures 5.6 to 5.8, all others figures are for one interferer.

Figure 5.3 shows the BEP versus SNR for DPSK at SIR $= -6$ dB. Curves labeled "Simulation" represent simulation results, while curves labeled "Analysis" show analytical results as yielded by the approximate upper bounds (5.60) (for $K = 2$)

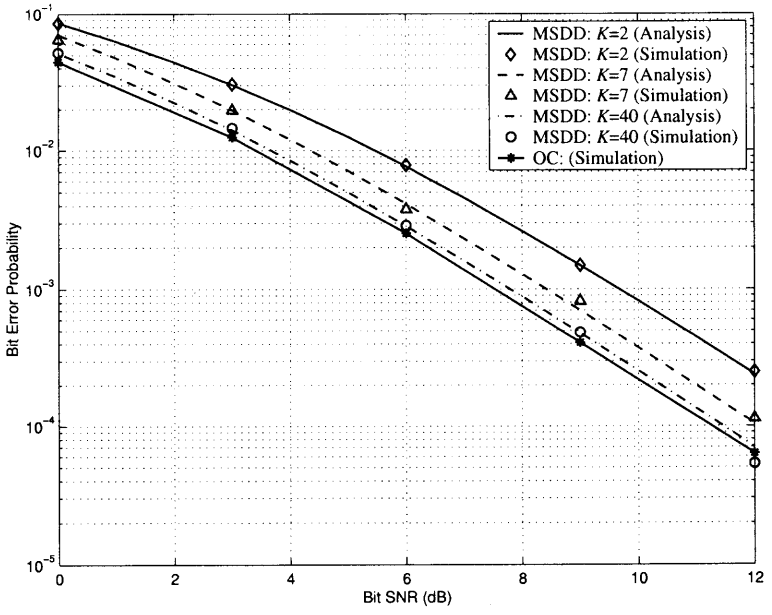

Figure 5.4 BEP versus SNR for $N_A = 4$ branches, $N_I = 1$ interferer, SIR $= -6$ dB, DQPSK modulation.

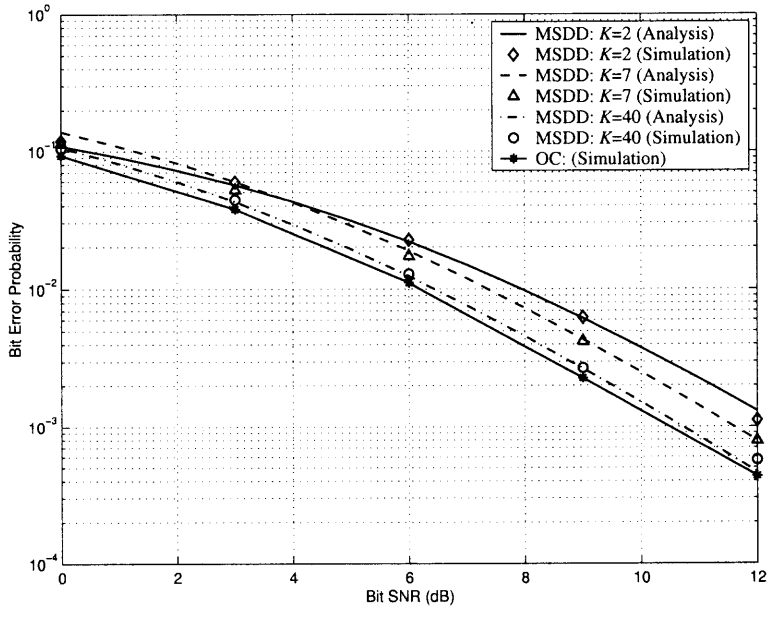

Figure 5.5 BEP versus SNR for $N_A = 4$ branches, $N_I = 1$ interferer, SIR $= -6$ dB, 8-DPSK modulation.

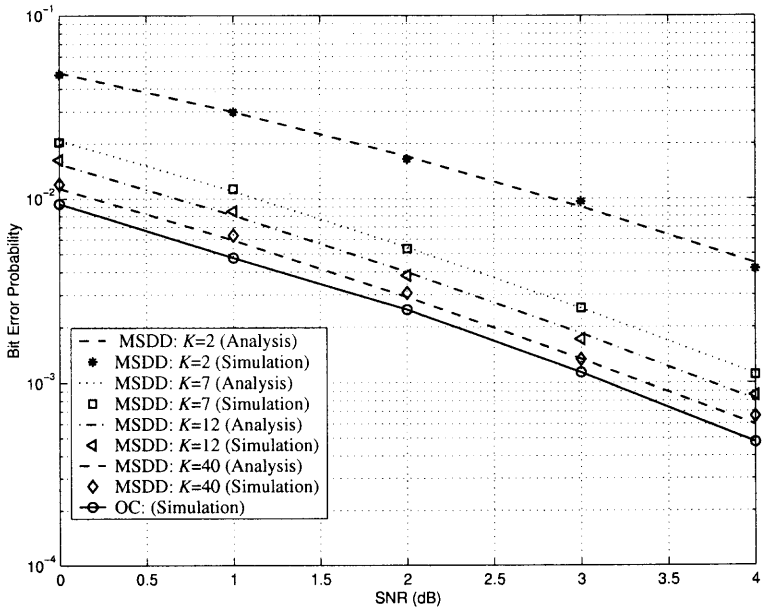

Figure 5.6 BEP versus SNR for $N_A = 8$ branches, $N_I = 3$ interferers, SIR $= -10$ dB, DPSK modulation.

and (5.62) (for $K > 2$). In all cases, PEP's were computed by (5.42) and (5.38). The interference plus noise term was generated such that its covariance matrix followed (5.3). The OC curve was generated by simulation. It can be observed that analysis results are very close to simulation results. It is also observed that the performance of MSDD approaches that of OC with differential encoding as K (the number of symbols in observation interval) increases. For example, at BEP $= 2 \times 10^{-3}$, when $K = 2$, the SNR difference between MSDD and OC is about 2.2 dB. When $K = 7$, the difference is about 1.0 dB. At $K = 40$, the difference becomes an insignificant 0.2 dB.

Figures 5.4 and 5.5 are respectively, for DQPSK and 8-DPSK. The curves in these figures follow the same trends as in Figure 5.3.

Figure 5.6 is for $N_A = 8$ branches, $N_I = 3$ interferers, and SIR $= -10$ dB. The analysis results are still very close to simulation results since SIR$\ll 1$. In Figure 5.7 where SIR $= 0$ dB, the analysis results are not close to simulation results. They are more like the upper bound of the simulation results. In Figure 5.8 only simulation

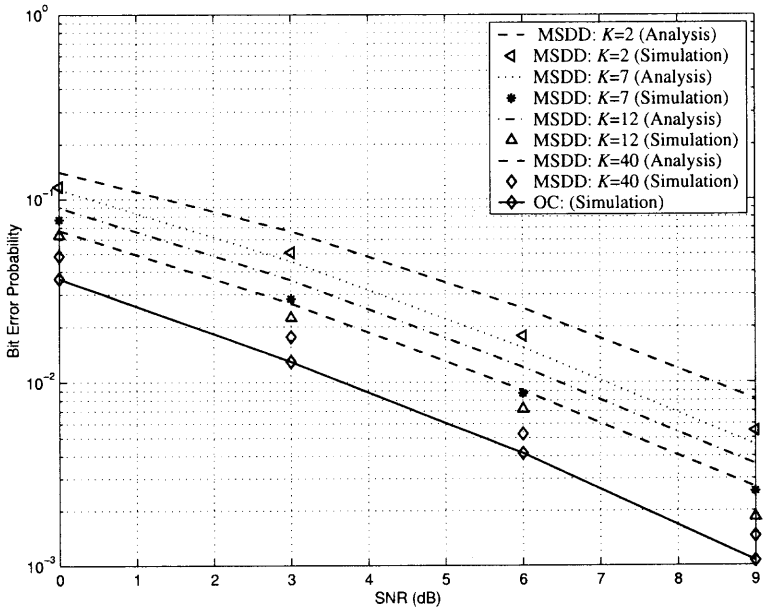

Figure 5.7 BEP versus SNR for $N_A = 6$ branches, $N_I = 4$ interferers, SIR = 0 dB, DPSK modulation.

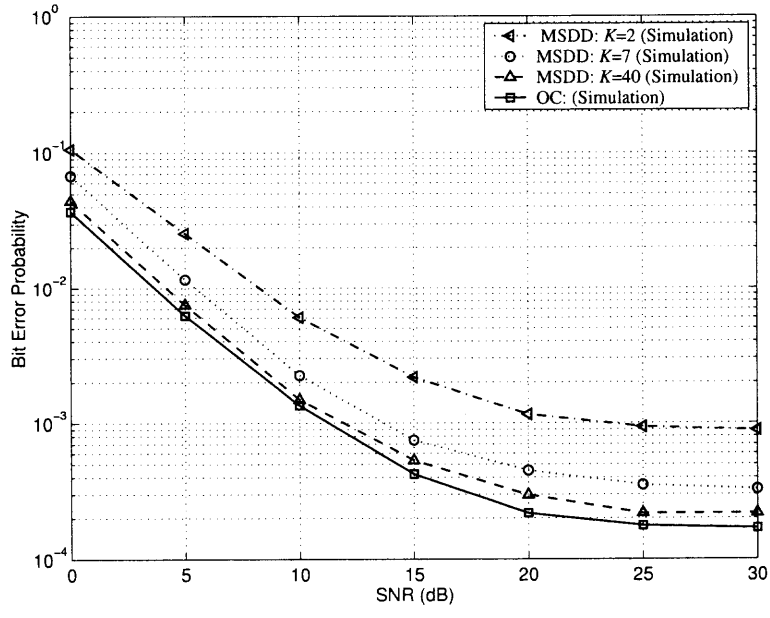

Figure 5.8 BEP versus SNR for $N_A = 4$ branches, $N_I = 4$ interferers, SIR = 10 dB, DPSK modulation.

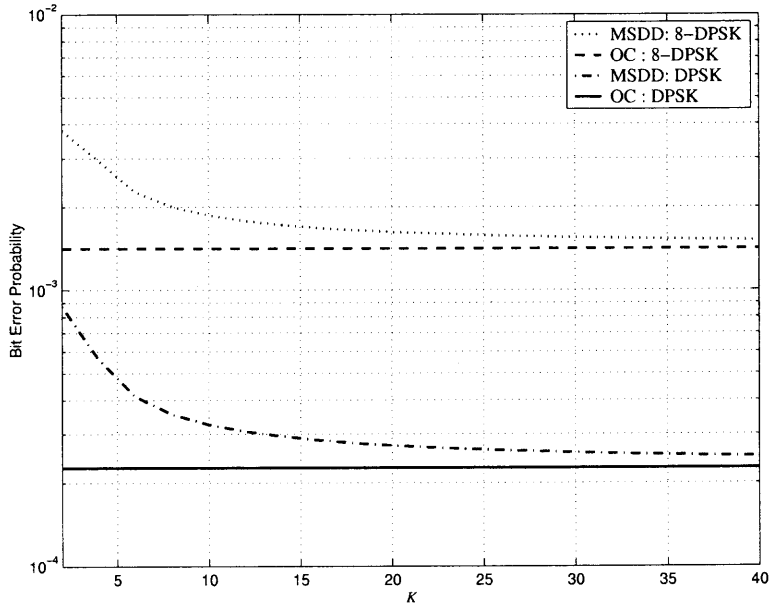

Figure 5.9 BEP versus the number of symbols in the observation interval K for $N_A = 4$ branches, $N_I = 1$ interferer, SIR $= -6$ dB.

results are shown. From Figures 5.6 to 5.8 corresponding to the case of more than one interferer, it can still be observed that the BEP of MSDD approaches that of OC as K increases.

The results shown in Figures 5.9 to 5.11 are all analytical results. In these figures, bit error probabilities are represented by their approximate upper bounds. The approximate upper bound is computed based on the PEP expressions (5.38) and (5.42), except for Figure 5.11.

Figure 5.9 shows the BEP of MSDD as a function of the number of symbols in the observation interval, K. It is evident that for both DPSK (binary modulation) and for 8-DPSK ($M = 8$), the performance of MSDD approaches that of OC as the observation interval increases.

Figure 5.10 shows the BEP versus SIR, for bit SNR $= 10$ dB, and for the cases of $K = 2$ and $K = 40$ symbols. When $K = 40$, MSDD achieves performance close to that of OC with differential encoding regardless of the SIR.

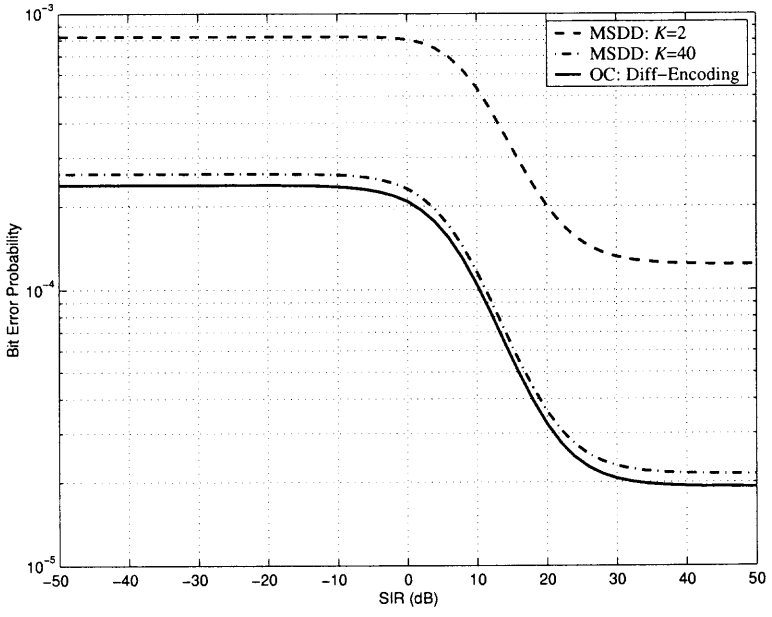

Figure 5.10 BEP versus SIR for $N_A = 4$ branches, $N_I = 1$ interferer, bit SNR = 10 dB, DQPSK modulation.

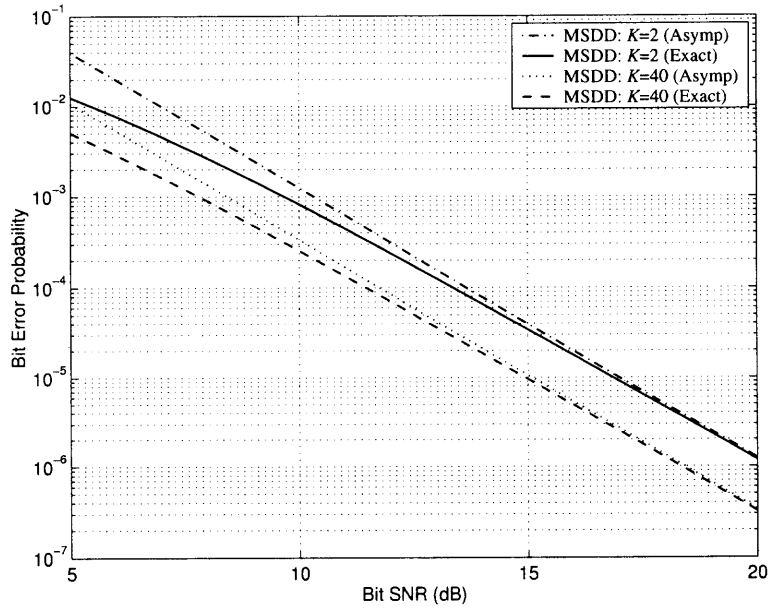

Figure 5.11 Comparison of asymptotic results and exact results for $N_A = 4$ branches, $N_I = 1$ interferer, SIR = -6 dB, DQPSK modulation.

Figure 5.11 is intended to verify the asymptotic large SNR approximation to the PEP. The signal modulation is DQPSK. Curves labeled "asymp" represent asymptotic results computed by applying (5.69) (for $K = 2$) and (5.70) (for $K = 40$); curves labeled "exact" represent exact results from (5.65) (for $K = 2$) and (5.66) (for $K = 40$). It is observed that for most SNR of interest (SNR > 10), the approximate upper bound based on asymptotic PEP is very close to the approximate upper bound based on the exact PEP.

CHAPTER 6

MSDD WITH UNKNOWN INTERFERENCE PHASE

6.1 Introduction

Two kinds of detectors for communication systems with reception diversity in the presence of white Gaussian noise and interference source have been discussed thus far. In Chapters 2 to 4, a detector using OC was discussed. To implement OC, side information on the channel gain of the desired signal and the covariance matrix of the interference plus noise must be available to the receiver. In Chapter 5, MSDD was discussed for the case where the channel gain of the desired signal was assumed to be unknown, but the covariance matrix of the interference plus noise was assumed to be known. Both detectors show the ability to suppress interference.

It would be desirable to be able to suppress the interference without requiring any information about the interference. Unfortunately, that is impossible. In Chapter 2, the interference plus noise is modeled as

$$\mathbf{z}_k = \sqrt{P_I} \sum_{i=1}^{N_I} \mathbf{c}_i s_{i,k} + \mathbf{n}_k. \tag{6.1}$$

If no information about $\mathbf{c}_i$ (which is assumed to be Gaussian distributed) is available, conditioned on the interference signal $s_{i,k}$, the interference term $\sqrt{P_I} \sum_{i=1}^{N_I} \mathbf{c}_i s_{i,k}$ would be the same as Gaussian noise and could not be distinguished from the white Gaussian noise $\mathbf{n}_k$. Therefore, at least some information about the interference is required.

In this chapter, a detector is developed for the case in which the only required channel information is the amplitude of the channels of the interference. The scenario is similar to that in Chapter 5. But in addition to assuming that the channel gain of the desired signal is unknown, the phase of the channel of the interference is assumed to be unknown as well. The channel amplitude of the interference is assumed known.

Moreover, the interference is assumed to have the same MDPSK modulation as the desired signal. A maximum-likelihood sequence detector (MLSD) is formulated for the joint detection of the desired signal and the interference. Simulation is performed for DPSK modulation. Simulation results in terms of BEP versus SNR are provided and compared with the results obtained by other detection schemes.

It is shown that when the interference level is high, this MSDD technique can achieve better performance than detectors using OC (with differential encoding).

6.2 System Model

The system model used in this chapter is similar to that presented in Section 1.2, except that now it is assumed there is only one interferer. The output of the match filter is

$$r_{k,l} = \sqrt{P_s} c_l s_k + \sqrt{P_I} c_{I,l} s_{I,k} + n_{k,l}, \; l = 1, 2, \cdots, N_A. \tag{6.2}$$

The definitions of the variables are in Section 1.2. Both the desired signal s_k and the interference source $s_{I,k}$ are assumed to be M-DPSK symbols.

The signals in vector notation are

$$\mathbf{r}_k = \sqrt{P_s}\mathbf{c} s_k + \sqrt{P_I}\mathbf{c}_I s_{I,k} + \mathbf{n}_k, \tag{6.3}$$

where $\mathbf{r}_k = [r_{k,1}, r_{k,2}, \cdots, r_{k,N_A}]^T$; $\mathbf{c}, \mathbf{c}_I$ and $\mathbf{n}_k$ are vectors that are similarly defined as $\mathbf{r}_k$.

Assume both c_l and $c_{I,l}$ are zero-mean complex Gaussian random variables (Rayleigh fading), and that they are mutually independent. For convenience, define $c_l = \alpha_l e^{j\phi_l}, c_{I,l} = \alpha_{I,l} e^{j\phi_{I,l}}$, vector $\boldsymbol{\alpha}$= $[\alpha_1, \alpha_1, \cdots, \alpha_{N_A}]$. Vectors $\boldsymbol{\phi}$, $\boldsymbol{\alpha}_I$ and $\boldsymbol{\phi}_I$ are defined similarly to $\boldsymbol{\alpha}$. In this chapter, $\boldsymbol{\alpha}_I$ is assumed to be known, but $\boldsymbol{\alpha}$, $\boldsymbol{\phi}$ and $\boldsymbol{\phi}_I$ are assumed to be unknown.

Assume the covariance matrix of $\mathbf{n}_k$ is

$$\mathbf{R}_n = E\left[\mathbf{n}_k \mathbf{n}_k^H\right] = \text{diag}\left[\sigma_1^2, \sigma_2^2, \cdots, \sigma_{N_A}^2\right], \tag{6.4}$$

where σ_l^2 $(l = 1, \cdots, N_A)$ is the power of the noise on the l-th branch.

Consider a sequence of K symbols running from time $k-(K-1)$ to k. Assume the channels are static within the duration of this sequence. Using vector notation,

$$\underline{\mathbf{r}}_k = \sqrt{P_s}\mathbf{H}\mathbf{s}_k + \sqrt{P_I}\mathbf{H}_I\mathbf{s}_{I,k} + \underline{\mathbf{n}}_k, \tag{6.5}$$

where $\underline{\mathbf{r}}_k = \left[\mathbf{r}_{k-(K-1)}, \mathbf{r}_{k-(K-2)}, \cdots, \mathbf{r}_k\right]^T$. The channel matrix is $\mathbf{H} = \mathbf{I}_K \otimes \mathbf{c}$, where $\mathbf{I}_K$ is the identity matrix of rank K. Other quantities are defined as: $\mathbf{H}_I = \mathbf{I}_K \otimes \mathbf{c}_I$, $\mathbf{s}_k = \left[s_{k-(K-1)}, \cdots, s_k\right]^T$, $\mathbf{s}_{I,k} = \left[s_{I,k-(K-1)}, \cdots, s_{I,k}\right]^T$, $\underline{\mathbf{n}}_k = \left[\mathbf{n}_{k-(K-1)}, \cdots, \mathbf{n}_k\right]^T$.

Some other assumptions about the signals and channels will be provided in the derivation of the decision statistic.

6.3 Decision Statistic

In this section, the decision statistic is derived for MSDD with known channel gain for the interference $\boldsymbol{\alpha}_I$. MSDD is a form of MLSD. The decision is made after K symbols are transmitted and received. Conditioned on the channels $\mathbf{c}$ and $\mathbf{c}_I$, the coherent decision criterion for sequence detection is given by

$$(\widehat{\mathbf{s}}_k, \widehat{\mathbf{s}}_{I,k}) = \arg \max_{\mathbf{s}_k, \mathbf{s}_{I,k}} p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{s}_{I,k}, \mathbf{c}, \mathbf{c}_I). \tag{6.6}$$

Note that both the desired and the interference symbols are detected. The pair $(\mathbf{s}_k, \mathbf{s}_{I,k})$ that maximizes $p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{s}_{I,k}, \mathbf{c}, \mathbf{c}_I)$ is chosen as the detected symbols. When the only known channel information is $\boldsymbol{\alpha}_I$, the decision rule for MLSD is

$$(\widehat{\mathbf{s}}_k, \widehat{\mathbf{s}}_{I,k}) = \arg \max_{\mathbf{s}_k, \mathbf{s}_{I,k}} p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{s}_{I,k}, \boldsymbol{\alpha}_I). \tag{6.7}$$

Next the expression for $p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{s}_{I,k}, \boldsymbol{\alpha}_I)$ is derived.

At time k, the probability of $\mathbf{r}_k$ conditioned on $s_k, s_{I,k}, \mathbf{c}$, and $\mathbf{c}_I$ is

$$\begin{aligned} & p(\mathbf{r}_k|s_k, s_{I,k}, \mathbf{c}, \mathbf{c}_I) \\ = \; & \pi^{-N_A}|\mathbf{R}_n|^{-1}\exp\left\{-\left(\mathbf{r}_k - \sqrt{P_s}\mathbf{c}s_k - \sqrt{P_I}\mathbf{c}_I s_{I,k}\right)^H \mathbf{R}_n^{-1}\right. \\ & \left.\times\left(\mathbf{r}_k - \sqrt{P_s}\mathbf{c}s_k - \sqrt{P_I}\mathbf{c}_I s_{I,k}\right)\right\} \qquad (6.8) \\ = \; & \pi^{-N_A}|\mathbf{R}_n|^{-1}\exp\left\{-\sum_{l=1}^{N_A}\frac{\left|r_{k,l}-\sqrt{P_s}c_l s_k - \sqrt{P_I}c_{I,l}s_{I,k}\right|^2}{\sigma_l^2}\right\}. \qquad (6.9) \end{aligned}$$

Assume $s_k, s_{I,k}$ and $\mathbf{n}_k$ are independent. The conditional probability of $\underline{\mathbf{r}}_k$ is

$$\begin{aligned} & p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{s}_{I,k}, \mathbf{c}, \mathbf{c}_I) \\ = \; & \prod_{i=k-(K-1)}^{k} p(\mathbf{r}_i|s_i, s_{I,i}, \mathbf{c}, \mathbf{c}_I) \\ = \; & \pi^{-N_A K}|\mathbf{R}_n|^{-K}\exp\left\{-\sum_{i=k-(K-1)}^{k}\sum_{l=1}^{N_A}\frac{\left|r_{i,l}-\sqrt{P_s}c_l s_i - \sqrt{P_I}c_{I,l}s_{I,i}\right|^2}{\sigma_l^2}\right\}. \qquad (6.10) \end{aligned}$$

By denoting the exponent in (6.10) as A, and expanding it,

$$\begin{aligned} A \; \triangleq \; & -\sum_{i=k-(K-1)}^{k}\sum_{l=1}^{N_A}\frac{\left|r_{i,l}-\sqrt{P_s}c_l s_i - \sqrt{P_I}c_{I,l}s_{I,i}\right|^2}{\sigma_l^2} \\ = \; & -\sum_{l=1}^{N_A}\frac{1}{\sigma_l^2}\sum_{i=k-(K-1)}^{k}|r_{i,l}|^2 - \sum_{l=1}^{N_A}\frac{1}{\sigma_l^2}\left[KP_I\alpha_{I,l}^2 - \sqrt{P_I}y_{I,l}^* c_{I,l}\right. \\ & \left. - \sqrt{P_I}y_{I,l}c_{I,l}^*\right] - \sum_{l=1}^{N_A}\frac{1}{\sigma_l^2}\left[KP_s\alpha_l^2 + \left(\sqrt{P_sP_I}yc_{I,l}^* - \sqrt{P_s}y_l^*\right)c_l\right. \\ & \left. + \left(\sqrt{P_sP_I}y^* c_{I,l} - \sqrt{P_s}y_l\right)c_l^*\right], \qquad (6.11) \end{aligned}$$

where

$$y_l = \sum_{i=k-(K-1)}^{k} r_{i,l} s_i^* \tag{6.12}$$

$$y_{I,l} = \sum_{i=k-(K-1)}^{k} r_{i,l} s_{I,i}^* \tag{6.13}$$

$$y = \sum_{i=k-(K-1)}^{k} s_i s_{I,i}^*. \tag{6.14}$$

Note that $y_l, y_{I,l}$ and y are functions of $\mathbf{s}_k$ and/or $\mathbf{s}_{I,k}$.

Define

$$C_0 = \sum_{l=1}^{N_A} \frac{1}{\sigma_l^2} \sum_{i=k-(K-1)}^{k} |r_{i,l}|^2 \tag{6.15}$$

$$C_l = \frac{1}{\sigma_l^2} \left[K P_I \alpha_{I,l}^2 - \sqrt{P_I} y_{I,l}^* c_{I,l} - \sqrt{P_I} y_{I,l} c_{I,l}^* \right] \tag{6.16}$$

$$x_l = |x_l| e^{i\psi_l} = \sqrt{P_s P_I} y c_{I,l}^* - \sqrt{P_s} y_l^*, \tag{6.17}$$

then (6.11) becomes

$$A = -C_0 - \sum_{l=1}^{N_A} C_l - \sum_{l=1}^{N_A} \frac{1}{\sigma_l^2} \left[K P_s \alpha_l^2 + 2 |x_l| \alpha_l \cos(\phi_l + \psi_l) \right]. \tag{6.18}$$

Substitute it into (6.10). Then

$$p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{s}_{I,k}, \mathbf{c}, \mathbf{c}_I) \tag{6.19}$$

$$= D_0 \exp\left\{ - \sum_{l=1}^{N_A} C_l \right\} \tag{6.20}$$

$$\times \exp\left\{ - \sum_{l=1}^{N_A} \frac{1}{\sigma_l^2} \left[K P_s \alpha_l^2 + 2 |x_l| \alpha_l \cos(\phi_l + \psi_l) \right] \right\}, \tag{6.21}$$

where

$$D_0 = \pi^{-N_A K} |\mathbf{R}|^{-K} \exp\{-C_0\}. \tag{6.22}$$

As mentioned above, the channel $\mathbf{c}$ is assumed to be unknown. Eliminate $\mathbf{c}$ in $p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{s}_{I,k}, \mathbf{c}, \mathbf{c}_I)$ by integrating over it,

$$\begin{aligned} & p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{s}_{I,k}, \mathbf{c}_I) \\ = & \int p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{s}_{I,k}, \mathbf{c}, \mathbf{c}_I) p_{\mathbf{c}}(\mathbf{c})\, d\mathbf{c} \\ = & \int\int p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{s}_{I,k}, \mathbf{c}, \mathbf{c}_I) p_{\boldsymbol{\alpha}}(\boldsymbol{\alpha})\, p_{\boldsymbol{\phi}}(\boldsymbol{\phi})\, d\boldsymbol{\alpha} d\boldsymbol{\phi}, \end{aligned} \tag{6.23}$$

where $p_{\mathbf{c}}(\mathbf{c}), p_{\boldsymbol{\alpha}}(\boldsymbol{\alpha})$ and $p_{\boldsymbol{\phi}}(\boldsymbol{\phi})$ are the PDFs of $\mathbf{c}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\phi}$ respectively.

Since the channels are independent of each other,

$$p_{\boldsymbol{\alpha}}(\boldsymbol{\alpha})\, p_{\boldsymbol{\phi}}(\boldsymbol{\phi}) = \prod_{l=1}^{N_A} p_{\alpha_l}(\alpha_l) p_{\phi_l}(\phi_l), \tag{6.24}$$

where $p_{\alpha_l}(\alpha_l)$ and $p_{\phi_l}(\phi_l)$ are the probability density functions of α_l and ϕ_l respectively. For Rayleigh fading channels,

$$p_{\alpha_l}(\alpha_l) = \frac{2\alpha_l}{\Omega_l} \exp\left(-\frac{\alpha_l^2}{\Omega_l}\right) \quad 0 \le \alpha_l < \infty \tag{6.25}$$

$$p_{\phi_l}(\phi_l) = \frac{1}{2\pi} \qquad 0 \le \phi_l < 2\pi, \tag{6.26}$$

where $\Omega_l = E[\alpha_l^2]$ is the mean square of the amplitude of the l-th channel.

Substitute (6.24), (6.25) and (6.26) into (6.23). After some straightforward manipulations,

$$\begin{aligned} & p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{s}_{I,k}, \mathbf{c}_I) \\ = & D_0 \prod_{l=1}^{N_A} \frac{\sigma_l^2}{KP_s\Omega_l + \sigma_l^2} \exp\left(\frac{\Omega_l |x_l|^2}{\sigma_l^2 (KP_s\Omega_l + \sigma_l^2)} - C_l\right). \end{aligned} \tag{6.27}$$

By expanding the exponent in (6.27) and collecting the terms that contain $\alpha_{I,l}$ and $\phi_{I,l}$,

$$\begin{aligned} & \frac{\Omega_l |x_l|^2}{\sigma_l^2 (KP_s\Omega_l + \sigma_l^2)} - C_l \\ = & D_{l,1} - \frac{KP_I}{\sigma_l^2}\alpha_{I,l}^2 + D_{l,2}\alpha_{I,l}^2 + 2|D_{l,3}|\,\alpha_{I,l}\cos(\phi_{I,l} - \varphi_l), \end{aligned} \tag{6.28}$$

where

$$D_{l,1} = \frac{\Omega_l P_s |y_l^*|^2}{\sigma_l^2 (K P_s \Omega_l + \sigma_l^2)} \tag{6.29}$$

$$D_{l,2} = \frac{\Omega_l P_s P_I |y|^2}{\sigma_l^2 (K P_s \Omega_l + \sigma_l^2)} \tag{6.30}$$

$$\begin{aligned} D_{l,3} &= |D_{l,3}| e^{i\varphi_l} \\ &= \frac{(K P_s \Omega_l + \sigma_l^2)\sqrt{P_I}}{\sigma_l^2 (K P_s \Omega_l + \sigma_l^2)} y_{I,l} - \frac{\Omega_l P_s \sqrt{P_I}}{\sigma_l^2 (K P_s \Omega_l + \sigma_l^2)} y y_l. \end{aligned} \tag{6.31}$$

Substitute (6.28) into (6.27),

$$\begin{aligned} & p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{s}_{I,k}, \mathbf{c}_I) \\ = \; & D_0 \prod_{l=1}^{N_A} \frac{\sigma_l^2}{K P_s \Omega_l + \sigma_l^2} \\ & \times \exp\left\{ D_{l,1} - \frac{K P_I}{\sigma_l^2} \alpha_{I,l}^2 + D_{l,2} \alpha_{I,l}^2 + 2 |D_{l,3}| \alpha_{I,l} \cos(\phi_{I,l} - \varphi_l) \right\}. \end{aligned} \tag{6.32}$$

In (6.32), the probability of $\underline{\mathbf{r}}_k$ is dependent on $\phi_{I,l}$ $(l = 1, \cdots, N_A)$ (which is included in $\mathbf{c}_I$). This dependence can be eliminated by integration,

$$p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{s}_{I,k}, \boldsymbol{\alpha}_I) = \int p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{s}_{I,k}, \mathbf{c}_I) p_{\boldsymbol{\phi}_I}(\boldsymbol{\phi}_I) \, d\boldsymbol{\phi}_I, \tag{6.33}$$

where $p_{\boldsymbol{\phi}_I}(\boldsymbol{\phi}_I)$ is the probability density function of $\boldsymbol{\phi}_I$. For independent Rayleigh fading channels,

$$p_{\boldsymbol{\phi}_I}(\boldsymbol{\phi}_I) = \left(\frac{1}{2\pi}\right)^{N_A} \quad 0 \le \phi_{I,1}, \cdots, \phi_{I,N_A} < 2\pi. \tag{6.34}$$

Substitute (6.32) and (6.34) into (6.33) and carry out the integration,

$$\begin{aligned} & p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{s}_{I,k}, \boldsymbol{\alpha}_I) \\ = \; & D_0 \left\{ \prod_{l=1}^{N_A} \frac{\sigma_l^2}{K P_s \Omega_l + \sigma_l^2} \right\} \exp\left\{ - \sum_{l=1}^{N_A} \frac{K P_I}{\sigma_l^2} \alpha_{I,l}^2 \right\} \eta_{N_A SDD}(\mathbf{s}_k, \mathbf{s}_{I,k}), \end{aligned} \tag{6.35}$$

where

$$\eta_{MSDD}(\mathbf{s}_k, \mathbf{s}_{I,k}) = \prod_{l=1}^{N_A} \exp\left\{ D_{l,1} + D_{l,2} \alpha_{I,l}^2 \right\} I_0(2 |D_{l,3}| \alpha_{I,l}), \tag{6.36}$$

and $I_0(x)$ is the zeroth order modified Bessel function of the first kind. Note that $\eta_{MSDD}\left(\mathbf{s}_k, \mathbf{s}_{I,k}\right)$ is a function of $\mathbf{s}_k$ and $\mathbf{s}_{I,k}$.

In (6.35), only $\eta_{MSDD}\left(\mathbf{s}_k, \mathbf{s}_{I,k}\right)$ is dependent on $\left(\mathbf{s}_k, \mathbf{s}_{I,k}\right)$. Maximizing $p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{s}_{I,k}, \boldsymbol{\alpha}_I)$ with respect to $\left(\mathbf{s}_k, \mathbf{s}_{I,k}\right)$ is equivalent to maximizing $\eta_{MSDD}\left(\mathbf{s}_k, \mathbf{s}_{I,k}\right)$. Therefore $\eta_{MSDD}\left(\mathbf{s}_k, \mathbf{s}_{I,k}\right)$ can be used as the decision statistic for the MSDD detector. The corresponding decision rule is

$$\left(\widehat{\mathbf{s}}_k, \widehat{\mathbf{s}}_{I,k}\right) = \arg \max_{\mathbf{s}_k, \mathbf{s}_{I,k}} \eta_{MSDD}\left(\mathbf{s}_k, \mathbf{s}_{I,k}\right). \tag{6.37}$$

The MSDD detector searches through all possible $\left(\mathbf{s}_k, \mathbf{s}_{I,k}\right)$ and chooses the pair that has the largest $\eta_{MSDD}\left(\mathbf{s}_k, \mathbf{s}_{I,k}\right)$ as the detected output.

To complete this section, the maximum-likelihood (ML) detector is introduced for the case when both $\mathbf{c}$ and $\mathbf{c}_I$ are known to the receiver. It is used in the simulation results section for comparison with MSDD. The ML detector is a kind of coherent detection technique that requires the channel gain of both the desired signal and interference. It makes symbol-by-symbol detection instead of sequence detection for MSDD. The ML decision rule is given by

$$\left(\widehat{s}_k, \widehat{s}_{I,k}\right) = \arg \max_{s_k, s_{I,k}} p(\mathbf{r}_k|s_k, s_{I,k}, \mathbf{c}, \mathbf{c}_I). \tag{6.38}$$

From (6.38) and $p(\mathbf{r}_k|s_k, s_{I,k}, \mathbf{c}, \mathbf{c}_I)$ shown in (6.9), the equivalent ML decision rule can be obtained as

$$\left(\widehat{s}_k, \widehat{s}_{I,k}\right) = \arg \max_{s_k, s_{I,k}} \eta_{ML}\left(s_k, s_{I,k}\right), \tag{6.39}$$

where the decision statistic

$$\eta_{ML}\left(s_k, s_{I,k}\right) = \sum_{l=1}^{N_A} \frac{\left|r_{k,l} - \sqrt{P_s} c_l s_k - \sqrt{P_I} c_{I,l} s_{I,k}\right|^2}{\sigma_l^2}. \tag{6.40}$$

6.4 Simulation Results

In the simulations, 4 receive branches and DPSK modulation were used. The channels of both the desired signal and the interference were assumed to have uniform power profiles, i.e., $\Omega_l = 1$, $E\left[\boldsymbol{\alpha}_{I,l}^2\right] = 1$ and $\sigma_l^2 = \sigma^2$ for $l = 1, 2, \cdots, N_A$.

In the figures SNR $= P_s/\sigma^2$, and SIR $= P_s/P_I$. Figure 6.1 and Figure 6.2 were generated for SIR $= 10$ dB and SIR $= -10$ dB, respectively.

The curves labeled "MSDD($K = 2$)" and "MSDD($K = 7$)" are the results for the MSDD detector developed in this chapter. It is observed that performance improves with the increase in K, the number of symbols in the observation interval. For example, in Figure 6.2, at BEP $= 2 \times 10^{-3}$, the required SNR for $K = 2$ is about 8.5 dB; for $K = 7$ it is 5.5 dB. That means increasing the observation interval from $K = 2$ to $K = 7$ symbols results in a 3 dB SNR improvement.

The curves labeled "OC" are the results for OC. The curves labeled "MSDD (known cov, $K = 13$)" are for the MSDD detector discussed in Chapter 5, which was developed for known covariance matrix of the interference plus noise. MSDD ($K = 7$) has about the same computation complexity as MSDD (known cov, $K = 13$).

In Figure 6.1, the BEP of MSDD ($K = 7$) is larger than that of MSDD (known cov, $K = 13$) and OC. In Figure 6.2, the BEP of MSDD ($K = 7$) is less than that of MSDD (known cov, $K = 13$) and OC. It can be concluded that at a high interference level, MSDD ($K = 7$) has better performance than MSDD (known cov, $K = 13$) and OC. This can be explained by the following: MSDD ($K = 7$) detects the interference signal $\mathbf{s}_{I,k}$ as well as the desired signal $\mathbf{s}_k$, but the MSDD (known cov, $K = 13$) and the OC detector detect only the desired signal. MSDD developed in this chapter is a kind of multiuser detection that gets better performance as the interference power increases. These results are reflected in Figure 6.3 which shows the difference between the required SNR of MSDD and OC at BEP$= 10^{-3}$.

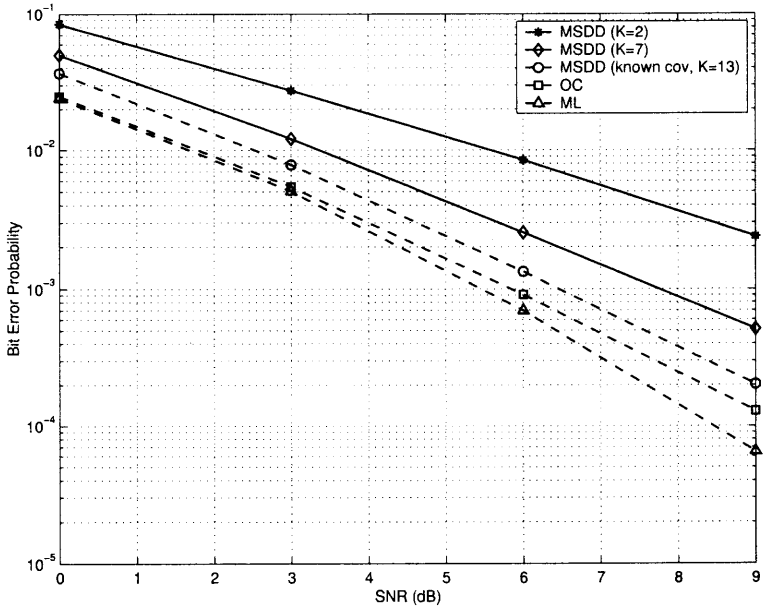

Figure 6.1 BEP versus SNR for MSDD, OC and ML. For $N_A = 4$ branches, SIR = 10 dB.

In both Figure 6.1 and Figure 6.2, the performance of MSDD ($K = 7$) is not as good as that of the maximum-likelihood detector (curves labeled "ML"). But the difference is small for a high interference level. In Figure 6.2, at BEP $= 2 \times 10^{-4}$, the difference of the required SNR is about 1.6 dB. These results are reflected in Figure 6.4 which shows the difference between the required SNR of MSDD and ML at BEP$= 10^{-3}$. It can be expected that this difference will decrease as K increases.

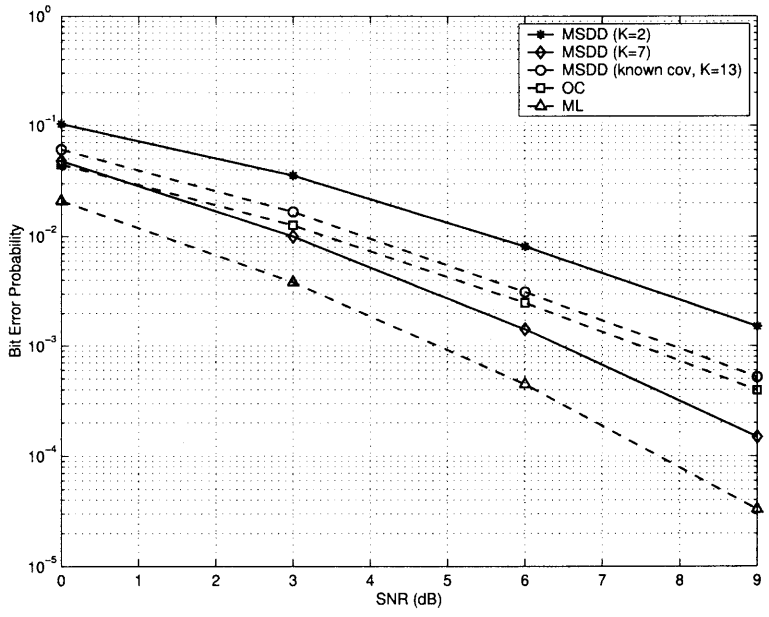

Figure 6.2 BEP versus SNR for MSDD, OC and ML. For $N_A = 4$ branches, SIR $= -10$ dB.

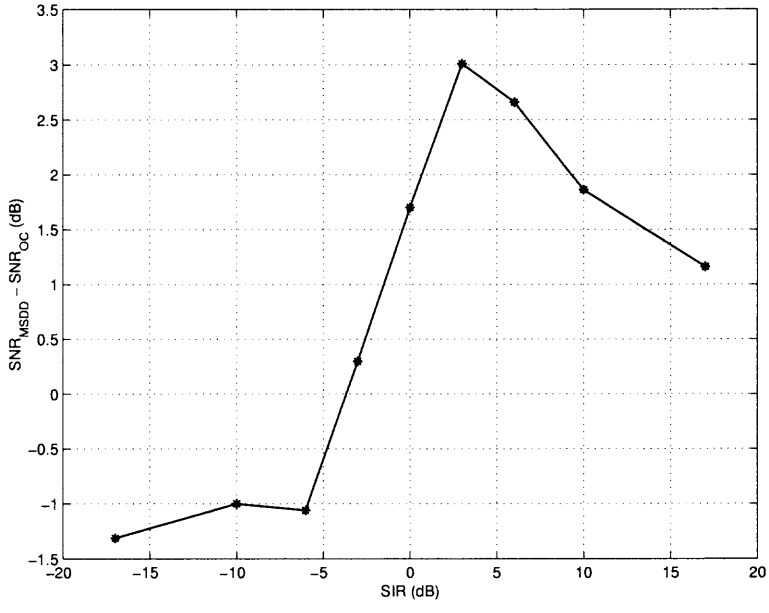

Figure 6.3 The difference between the required SNR (at BEP= 10^{-3}) of MSDD and OC versus SIR.

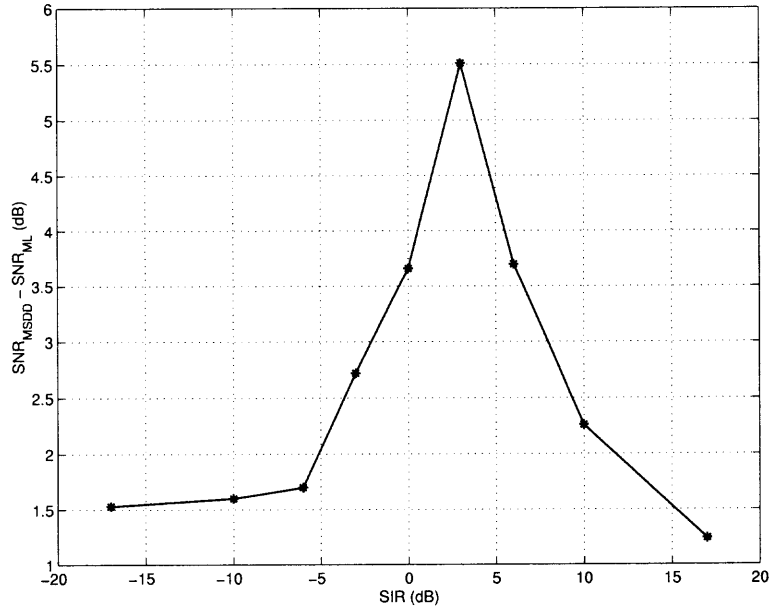

Figure 6.4 The difference between the required SNR (at BEP= 10^{-3}) of MSDD and ML versus SIR.

CHAPTER 7

SUMMARY AND SUGGESTIONS FOR FUTURE WORK

7.1 Summary

This dissertation accomplished the following:

- Obtained closed-form expressions of the BEP of OC with BPSK modulation.
- Derived expressions of the SEP and BEP of OC with M-PSK modulation, which only involve integration over elementary functions.
- Formulated simpler asymptotic expressions of BEP for OC with M-PSK modulation.
- Derived expressions of the SEP of OC with M-PSK modulation. The expressions can be used for systems with interference of unequal power levels.
- Developed the decision statistic of MSDD. The performance of this detection scheme was analyzed. Through analysis results and simulation results, it is proven that with an increasing observation interval, the performance of MSDD approaches that of OC with differential encoding.
- Evaluated the performance of MSDD for the case when the channel gain information of the interference is known, but the phase is unknown.

To summarize, for OC, there are three approaches for analyzing the error probability: the first starts from the decision metric, the second starts from the MGF of SINR, and the third starts from the reliability function. All approaches lead to closed-form expressions for the case of BPSK modulation. Since the reliability function is in simple form and can be used for systems with unequal interference power levels, the approach presented in Chapter 4 is less complicated than other

approaches, and its final expressions are relatively less involved and more desirable. For MSDD, two types of detection schemes have been developed: one for systems with a known covariance matrix of interference plus noise, and one for systems with known interference amplitude but unknown phase. Both show good performance with the increase of the number of symbols in the observation interval. The MSDD scheme presented in Chapter 5 is practical for long observation intervals after the introduction of an iterative decision feedback algorithm, which greatly reduces the computational complexity.

7.2 Suggestions for Future Work

Looking into the future, there are three research topics related to this dissertation:

- Evaluate the error probability of OC and MSDD for a more realistic channel

Thus far, all the analyses and simulations are based on the assumption that the channel information (amplitude, phase, covariance matrix, etc.) is perfectly known at the receiver. In future work, more realistic channel models and information will be considered. For example, the channel gain and covariance matrix will be estimated through training sequences. The MSDD, which was developed on the assumption that the channel is static within the observation interval, should be evaluated for the case in which the channel is time-varying at various rates. Work could also be done for developing MSDD algorithms for other fading channel models (such as Ricean, Nakagami, etc.), or for applying the algorithms developed for Rayleigh channels to other channel models in order to evaluate the performance. New expressions may be derived for OC applying to various channel models. With modifications, some methods presented in this dissertation may be applied to systems with both transmit diversity and receive diversity.

- Evaluate the error probability of OC and MSDD for more complicated system models

The discussion in the previous chapters focuses on simplified systems with M-PSK or M-DPSK modulation. Other modulation schemes, such as quadrature amplitude modulation (QAM), can be evaluated in future work. OC and MSDD can be incorporated into more complicated systems with coding.

- Evaluate the SINR for MSDD

The SINR of OC has been studied thoroughly and good expressions have been derived. In some applications, SINR is a more convenient parameter than error probability, especially when it comes to analyzing the performance of systems with many modules. For MSDD, all the research until now has been focused on error probability analysis. The analysis of output SINR of MSDD will facilitate its application to more complex systems.

APPENDIX A

DERIVATION OF THE CHARACTERISTIC FUNCTION FOR OC

In this appendix, the expression for the characteristic functions $\Phi_{D|\boldsymbol{\lambda}}(j\omega)$ of the test statistic D of OC is derived.

Define

$$d_m = \lambda_m^{-1} g_m^* x_m + \lambda_m^{-1} g_m x_m^*, \tag{A.1}$$

then from (2.5),

$$D = \sum_{m=1}^{N_A} d_m. \tag{A.2}$$

From the signal model in Section 2.2, and the definition of the whitened interference-plus-noise vector $\mathbf{x}$ and the modified channel vector $\mathbf{g}$ in (2.6) and (2.7), the covariance matrix of $\mathbf{x}$ and $\mathbf{g}$ can be evaluated as

$$\mathbf{R}_{xx} = E\left[\mathbf{x}\mathbf{x}^H\right] = P_s \mathbf{I}_{N_A} + \boldsymbol{\Lambda} \tag{A.3}$$

$$\mathbf{R}_{gg} = \mathbf{I}_{N_A} \tag{A.4}$$

$$\mathbf{R}_{xg} = \mathbf{R}_{gx} = \sqrt{P_s}\mathbf{I}_{N_A}. \tag{A.5}$$

To use the results in [2, Appendix B], identify the following quantities using the notation in the reference: $X_m = x_m$, $Y_m = g_m$. Then from (A.3) to (A.5) (in the notation of the reference)

$$\overline{X}_m = \overline{Y}_m = 0 \tag{A.6}$$

$$\mu_{xx,m} = \frac{1}{2}(P_s + \lambda_m) \tag{A.7}$$

$$\mu_{yy,m} = \frac{1}{2} \tag{A.8}$$

$$\mu_{xy,m} = \mu_{yx,m} = \frac{1}{2}\sqrt{P_s}. \tag{A.9}$$

Substitute the above equations into Eq. (B-6) and Eq. (B-5) in [2, Appendix B], together with $A_m = B_m = 0$ and $C_m = \lambda_m^{-1}$. After some straightforward manipulations, the characteristic function of d_m is obtained as

$$\phi_{d_m}(j\omega) = \frac{\lambda_m}{\left[\omega + j\left(\sqrt{P_s + \lambda_m} - \sqrt{P_s}\right)\right]\left[\omega - j\left(\sqrt{P_s + \lambda_m} + \sqrt{P_s}\right)\right]}. \tag{A.10}$$

By substituting (A.10) into $\Phi_{D|\boldsymbol{\lambda}}(j\omega) = \prod_{m=1}^{N_A} \phi_{d_m}(j\omega)$ and using $\lambda_{N_{\min}+1} = \lambda_{N_{\min}+2} = \cdots = \lambda_{N_A} = \sigma^2$, $\Phi_{D|\boldsymbol{\lambda}}(j\omega)$ is obtained as shown in (2.9).

APPENDIX B

EVALUATION OF THE RESIDUES

In this appendix, the residue shown in (2.8) is evaluated.

By using (2.9), the term $\Phi_{D|\boldsymbol{\lambda}}(j\omega)/\omega$ in (2.8) can be expressed as

$$\frac{\Phi_{D|\boldsymbol{\lambda}}(j\omega)}{\omega} = \frac{1}{\omega}\left[\frac{\omega_1\omega_2}{(\omega-\omega_1)(\omega-\omega_2)}\right]^{N_A-N_{\min}}\prod_{m=1}^{N_{\min}}\frac{\omega_{1,m}\omega_{2,m}}{(\omega-\omega_{1,m})(\omega-\omega_{2,m})}. \tag{B.1}$$

Since $\mathrm{Im}(\omega_1) < 0, \mathrm{Im}(\omega_{1,m}) < 0, \mathrm{Im}(\omega_2) > 0,$ and $\mathrm{Im}(\omega_{2,m}) > 0,$

$$\sum_{\mathrm{Im}(\omega_m)>0}\mathrm{Res}\left[\frac{\Phi_{D|\boldsymbol{\lambda}}(j\omega)}{\omega};\omega_m\right] = \sum_{m=1}^{N_{\min}}\mathrm{Res}\left[\frac{\Phi_{D|\boldsymbol{\lambda}}(j\omega)}{\omega};\omega_{2,m}\right] + \mathrm{Res}\left[\frac{\Phi_{D|\boldsymbol{\lambda}}(j\omega)}{\omega};\omega_2\right]. \tag{B.2}$$

The residues are evaluated using the following complex variables theory [49]: if a function $f(\omega)$ has a pole ω_0 of order N, the residue of $f(\omega)$ at ω_0 is

$$\mathrm{Res}\left[f(\omega);\omega_0\right] = \frac{1}{(N-1)!}\frac{d^{(N-1)}}{d\omega^{(N-1)}}\left[(\omega-\omega_0)^N f(\omega)\right]\Big|_{\omega=\omega_0}. \tag{B.3}$$

B.1 Evaluation of the Residues at Poles $\omega_{2,m}$

For $1 \le m \le N_{\min}$, the poles $\omega_{2,m}$ are of order 1, hence

$$\begin{aligned}
&\mathrm{Res}\left[\frac{\Phi_{D|\boldsymbol{\lambda}}(j\omega)}{\omega};\omega_{2,m}\right] \\
&= (\omega-\omega_{2,m})\frac{\Phi_{D|\boldsymbol{\lambda}}(j\omega)}{\omega}\Big|_{\omega=\omega_{2,m}} \\
&= \frac{1}{\omega_{2,m}}\left[\frac{\omega_1\omega_2}{(\omega_{2,m}-\omega_1)(\omega_{2,m}-\omega_2)}\right]^{N_A-N_{\min}}\frac{\omega_{1,m}\omega_{2,m}}{(\omega_{2,m}-\omega_{1,m})} \\
&\quad\times\prod_{n=1,n\neq m}^{N_{\min}}\frac{\omega_{1,n}\omega_{2,n}}{(\omega_{2,m}-\omega_{1,n})(\omega_{2,m}-\omega_{2,n})} \\
&= \left[\frac{\omega_1\omega_2}{(\omega_{2,m}-\omega_1)(\omega_{2,m}-\omega_2)}\right]^{N_A-N_{\min}}\frac{\omega_{1,m}}{(\omega_{2,m}-\omega_{1,m})} \\
&\quad\times\prod_{n=1,n\neq m}^{N_{\min}}\frac{\omega_{1,n}\omega_{2,n}}{(\omega_{2,m}-\omega_{1,n})(\omega_{2,m}-\omega_{2,n})}.
\end{aligned} \tag{B.4}$$

B.2 Evaluation of the Residue at Pole ω_2

Since the pole $\omega = \omega_2$ is of order $(N_A - N_{\min})$,

$$\begin{aligned}
&\operatorname{Res}\left[\frac{\Phi_{D|\boldsymbol{\lambda}}(j\omega)}{\omega};\omega_2\right] \\
&= \frac{1}{(N_A - N_{\min} - 1)!}\frac{d^{N_A-N_{\min}-1}}{d\omega^{N_A-N_{\min}-1}}\left\{(\omega-\omega_2)^{N_A-N_{\min}}\frac{\Phi_{D|\boldsymbol{\lambda}}(j\omega)}{\omega}\right\}\Bigg|_{\omega=\omega_2} \\
&= \frac{1}{(N_A - N_{\min} - 1)!} \\
&\quad \times \frac{d^{N_A-N_{\min}-1}}{d\omega^{N_A-N_{\min}-1}}\left\{\left(\frac{\omega_1\omega_2}{\omega-\omega_1}\right)^{N_A-N_{\min}}\frac{1}{\omega}\prod_{m=1}^{N_{\min}}\frac{\omega_{1,m}\omega_{2,m}}{(\omega-\omega_{1,m})(\omega-\omega_{2,m})}\right\}\Bigg|_{\omega=\omega_2}.
\end{aligned} \tag{B.5}$$

First expand the production into summation as

$$\begin{aligned}
&\left(\frac{\omega_1\omega_2}{\omega-\omega_1}\right)^{N_A-N_{\min}}\frac{1}{\omega}\prod_{m=1}^{N_{\min}}\frac{\omega_{1,m}\omega_{2,m}}{(\omega-\omega_{1,m})(\omega-\omega_{2,m})} \\
&= \left(\frac{\omega_1\omega_2}{\omega-\omega_1}\right)^{N_A-N_{\min}}\left[\frac{A_0}{\omega}+\sum_{m=1}^{N_{\min}}\frac{A_{1,m}}{(\omega-\omega_{1,m})}+\sum_{m=1}^{N_{\min}}\frac{A_{2,m}}{(\omega-\omega_{2,m})}\right],
\end{aligned} \tag{B.6}$$

where

$$A_0 = \prod_{m=1}^{N_{\min}}\frac{\omega_{1,m}\omega_{2,m}}{(\omega-\omega_{1,m})(\omega-\omega_{2,m})}\Bigg|_{\omega=0} = 1 \tag{B.7}$$

$$\begin{aligned}
A_{1,m} &= \frac{1}{\omega}\frac{\omega_{1,m}\omega_{2,m}}{(\omega-\omega_{2,m})}\prod_{n=1,n\neq m}^{N_{\min}}\frac{\omega_{1,n}\omega_{2,n}}{(\omega-\omega_{1,n})(\omega-\omega_{2,n})}\Bigg|_{\omega=\omega_{1,m}} \\
&= \frac{\omega_{2,m}}{(\omega_{1,m}-\omega_{2,m})}\prod_{n=1,n\neq m}^{N_{\min}}\frac{\omega_{1,n}\omega_{2,n}}{(\omega_{1,m}-\omega_{1,n})(\omega_{1,m}-\omega_{2,n})}
\end{aligned} \tag{B.8}$$

$$\begin{aligned}
A_{2,m} &= \frac{1}{\omega}\frac{\omega_{1,m}\omega_{2,m}}{(\omega-\omega_{1,m})}\prod_{n=1,n\neq m}^{N_{\min}}\frac{\omega_{1,n}\omega_{2,n}}{(\omega-\omega_{1,n})(\omega-\omega_{2,n})}\Bigg|_{\omega=\omega_{2,m}} \\
&= \frac{\omega_{1,m}}{(\omega_{2,m}-\omega_{1,m})}\prod_{n=1,n\neq m}^{N_{\min}}\frac{\omega_{1,n}\omega_{2,n}}{(\omega_{2,m}-\omega_{1,n})(\omega_{2,m}-\omega_{2,n})}.
\end{aligned} \tag{B.9}$$

Then by substituting (B.6) into (B.5) and using the following derivation principles

$$\frac{d^n}{d\omega^n}\left[f(\omega)\,g(\omega)\right] = \sum_{l=0}^{n}\binom{n}{l}\frac{d^{n-l}f(\omega)}{d\omega^{n-l}}\frac{d^l g(\omega)}{d\omega^l} \tag{B.10}$$

$$\frac{d^n}{d\omega^n}\frac{1}{(\omega+a)^m} = (-1)^n\frac{(m+n-1)!}{(m-1)!}\frac{1}{(\omega+a)^{m+n}}, \tag{B.11}$$

after some manipulations, it follows that

$$\begin{aligned}
&\mathrm{Res}\left[\frac{\Phi_{D|\boldsymbol{\lambda}}(j\omega)}{\omega};\omega_2\right] \\
&= \frac{(\omega_1\omega_2)^{N_A-N_{\min}}}{(N_A-N_{\min}-1)!}(-1)^{N_A-N_{\min}-1}\sum_{l=0}^{N_A-N_{\min}-1}\frac{(N_A-N_{\min}-1+l)!}{l!} \\
&\times\left\{\frac{1}{\omega_2^{N_A-N_{\min}-l}}+\sum_{m=1}^{N_{\min}}\left[A_{1,m}\frac{1}{(\omega_2-\omega_{1,m})^{N_A-N_{\min}-l}}+A_{2,m}\frac{1}{(\omega_2-\omega_{2,m})^{N_A-N_{\min}-l}}\right]\right\} \\
&\times\frac{1}{(\omega_2-\omega_1)^{N_A-N_{\min}+l}}.
\end{aligned} \tag{B.12}$$

B.3 Summary

By substituting (B.4) and (B.12) into (B.2),

$$\begin{aligned}
&\sum_{\mathrm{Im}(\omega_m)>0}\mathrm{Res}\left[\frac{\Phi_{D|\boldsymbol{\lambda}}(j\omega)}{\omega};\omega_m\right] \\
&= \sum_{m=1}^{N_{\min}}\left[\frac{\omega_1\omega_2}{(\omega_{2,m}-\omega_1)(\omega_{2,m}-\omega_2)}\right]^{N_A-N_{\min}}\frac{\omega_{1,m}}{(\omega_{2,m}-\omega_{1,m})} \\
&\times\prod_{n=1,n\neq m}^{N_{\min}}\frac{\omega_{1,n}\omega_{2,n}}{(\omega_{2,m}-\omega_{1,n})(\omega_{2,m}-\omega_{2,n})} \\
&+\frac{(\omega_1\omega_2)^{N_A-N_{\min}}}{(N_A-N_{\min}-1)!}(-1)^{N_A-N_{\min}-1}\sum_{l=0}^{N_A-N_{\min}-1}\frac{(N_A-N_{\min}-1+l)!}{l!}\left\{\frac{1}{\omega_2^{N_A-N_{\min}-l}}\right. \\
&\left.+\sum_{m=1}^{N_{\min}}\left[A_{1,m}\frac{1}{(\omega_2-\omega_{1,m})^{N_A-N_{\min}-l}}+A_{2,m}\frac{1}{(\omega_2-\omega_{2,m})^{N_A-N_{\min}-l}}\right]\right\} \\
&\times\frac{1}{(\omega_2-\omega_1)^{N_A-N_{\min}+l}}.
\end{aligned} \tag{B.13}$$

APPENDIX C

EVALUATION OF THE SUM OF INTEGRALS

In this appendix, the sum of integrals $\sum_{m=1}^{N_{\min}} \int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y}$ is evaluated and the relation in (2.42) is proven. The main steps of the evaluation are as follows:

1. Combine the sum of $N_{\min}$ integrals (each is an $N_{\min}$-fold integral) into one $N_{\min}$-fold integral.

2. Change the integration limits to simplify the evaluation of the integral.

3. For $N_{\min} > 1$, separate the $N_{\min}$-fold integration.

C.1 Combination of $N_{\min}$ Integrals

Here the goal is to convert the sum of $N_{\min}$ integrals into one integral. First consider the integrals $\int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y}$ for $1 < m < N_{\min}$, then for $m = 1$ and $m = N_{\min}$.

For each m, $1 < m < N_{\min}$, first carry out the integration over y_m, then over $y_{N_{\min}}$, next $y_{N_{\min}-1}$, and finally over y_1. Recall that the values of the y_m's (defined in (2.23)) descend with the increase in index m. The integration limits of y_m are $y_{m+1} \le y_m \le y_{m-1}$. Since $\eta \le y_{N_{\min}} \le \cdots \le y_2 \le y_1 < \infty$, $\int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y}$ can be expressed as

$$\begin{aligned} &\int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\ &= \int_{\eta}^{\infty} \left\{ \int_{\eta}^{y_1} \cdots \left\{ \int_{\eta}^{y_{m-2}} \left\{ \int_{\eta}^{y_{m-1}} \cdots \left\{ \int_{\eta}^{y_{N_{\min}-1}} \left\{ \int_{y_{m+1}}^{y_{m-1}} f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) dy_m \right\} dy_{N_{\min}} \right\} \right. \right. \right. \\ &\quad \left. \left. \left. \times \cdots dy_{m+1} \right\} dy_{m-1} \right\} \cdots dy_2 \right\} dy_1, \end{aligned} \tag{C.1}$$

with the inner integrations being carried out before the outer integrations.

Substituting $f_m(\mathbf{y})$ (from (2.28)) and $p_{\mathbf{y}}(\mathbf{y})$ (from (2.31)) into (C.1), and changing the variables from y to t as follows:

$$y_1 \to t_1,\ y_2 \to t_2, \cdots,\ y_{m-1} \to t_{m-1},\ y_{m+1} \to t_m, \cdots,\ y_{N_{\min}} \to t_{N_{\min}-1},\ y_m \to t_{N_{\min}},$$

an expression (which is omitted here to save space) of $\int f_m(\mathbf{y})\, p_{\mathbf{y}}(y)dy$ for each m, $1 < m < N_{\min}$ is obtained. Expressions for $m = 1$ and $m = N_{\min}$ can be obtained similarly. It can be shown that the sum of these expressions is

$$\begin{aligned}
&\sum_{m=1}^{N_{\min}} \int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y})d\mathbf{y} \\
= \; & \frac{K_1}{2} z_0^{N-N_{\min}} \int_\eta^\infty \int_\eta^{t_1} \cdots \int_\eta^{t_{N_{\min}-2}} \left\{ \prod_{n=1}^{N_{\min}-1} \left(1-t_n^2\right)\left(t_n^2-\eta^2\right)^{N_{\max}-N_{\min}} t_n \right. \\
& \left. \times \exp\left[-\beta\left(t_n^2-\eta^2\right)\right] \right\} \left[\prod_{1\le i<j\le N_{\min}-1} \left(t_i^2-t_j^2\right)^2 \right] \\
& \times \left\{ \int_\eta^\infty \left(1-t_{N_{\min}}\right)\left(t_{N_{\min}}^2-\eta^2\right)^{N_{\max}-N} \right. \\
& \left. \times \left[\prod_{n=1}^{N_{\min}-1} \left(t_{N_{\min}}^2-t_n^2\right) \right] \exp\left[-\beta\left(t_{N_{\min}}^2-\eta^2\right)\right] dt_{N_{\min}} \right\} dt_{N_{\min}-1} \cdots dt_2 dt_1.
\end{aligned} \tag{C.2}$$

where $z_0 = 1-\eta^2 = -1/\gamma$. Note that (C.2) consists of only one $N_{\min}$-fold integral.

To simplify notation, perform the change of variables: $(t_n^2-\eta^2) \rightarrow z_n$ for $n = 1$, $2, \cdots, N_{\min}-1$. Then (C.2) becomes

$$\begin{aligned}
&\sum_{m=1}^{N_{\min}} \int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y})d\mathbf{y} \\
= \; & \frac{K_1}{2^{N_{\min}}} z_0^{N-N_{\min}} \int_0^\infty \int_0^{z_1} \cdots \int_0^{z_{N_{\min}-2}} \left\{ \prod_{n=1}^{N_{\min}-1} \left(z_0-z_n\right) z_n^{N_{\max}-N_{\min}} \exp\left(-\beta z_n\right) \right\} \\
& \times \left[\prod_{1\le i<j\le N_{\min}-1} \left(z_i-z_j\right)^2 \right] \left\{ \int_\eta^\infty \left(1-t_{N_{\min}}\right)\left(t_{N_{\min}}^2-\eta^2\right)^{N_{\max}-N} \right. \\
& \left. \times \left[\prod_{n=1}^{N_{\min}-1} \left(t_{N_{\min}}^2-\eta^2-z_n\right) \right] \exp\left[-\beta\left(t_{N_{\min}}^2-\eta^2\right)\right] dt_{N_{\min}} \right\} dz_{N_{\min}-1} \cdots dz_2 dz_1.
\end{aligned} \tag{C.3}$$

C.2 Change of Integration Limits

Consider the integrations in (C.3). The integration limits for variables z_i ($1 \leq i \leq N_{\min} - 1$) are (listed in the order of integration starting with the innermost integral): $0 \leq z_{N_{\min}-1} \leq z_{N_{\min}-2}, \cdots, 0 \leq z_3 \leq z_2, 0 \leq z_2 \leq z_1, 0 \leq z_1 < \infty$. Make the following observations:

(1) The integrand is symmetric with respect to the variables z_i, $1 \leq i \leq N_{\min} - 1$ (i.e., for any $1 \leq i, j \leq N_{\min} - 1$, $i \neq j$, z_i and z_j can be interchanged leaving the integrand the same).

(2) There are $(N_{\min} - 1)!$ possible permutations of the integration limits of z_i for which (C.3) will yield the same result. For example, one such permutation is: $0 \leq z_{N_{\min}-1} \leq z_{N_{\min}-2}, \cdots, 0 \leq z_3 \leq z_1, 0 \leq z_1 \leq z_2, 0 \leq z_2 < \infty$. In this example, the order of integration is reversed for z_1 and z_2.

(3) These $(N_{\min} - 1)!$ integration limits are disjoint, and their union is the region: $0 \leq z_1 < \infty, 0 \leq z_2 < \infty, 0 \leq z_3 < \infty, \cdots, 0 \leq z_{N_{\min}-1} < \infty$.

It follows that (C.3) is equal to the integrand integrated over $0 \leq z_1, z_2, z_3, \cdots, z_{N_{\min}-1} < \infty$ and divided by $(N_{\min} - 1)!$:

$$
\begin{aligned}
& \sum_{m=1}^{N_{\min}} \int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\
= \; & \frac{1}{(N_{\min}-1)!} \frac{K_1}{2^{N_{\min}}} z_0^{N-N_{\min}} \\
& \times \int_0^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} \left\{ \prod_{n=1}^{N_{\min}-1} (z_0 - z_n)\, z_n^{N_{\max}-N_{\min}} \exp(-\beta z_n) \right\} \\
& \times \left[\prod_{1 \leq i < j \leq N_{\min}-1} (z_i - z_j)^2 \right] \left\{ \int_{\eta}^{\infty} (1 - t_{N_{\min}}) \left(t_{N_{\min}}^2 - \eta^2\right)^{N_{\max}-N} \right. \\
& \left. \times \left[\prod_{n=1}^{N_{\min}-1} \left(t_{N_{\min}}^2 - \eta^2 - z_n\right) \right] \exp\left[-\beta \left(t_{N_{\min}}^2 - \eta^2\right)\right] dt_{N_{\min}} \right\} \\
& \times dz_{N_{\min}-1} \cdots dz_2 dz_1.
\end{aligned}
\tag{C.4}
$$

The integration above is treated differently for $N_{\min} = 1$ and $N_{\min} > 1$.

For $N_{\min} = 1$, (C.4) is a single integration. Use (2.33) and $z_0 = -1/\gamma$,

$$\begin{aligned} & \sum_{m=1}^{N_{\min}} \int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\ = & \frac{1}{(N_{\max}-1)!} \beta^{N_{\max}} \left(-\frac{1}{\gamma}\right)^{N-1} \left(B_{N_{\max}-N} - \frac{1}{2}\frac{(N_{\max}-N)!}{\beta^{N_{\max}-N+1}}\right), \end{aligned} \tag{C.5}$$

where

$$B_q \triangleq \int_{\eta}^{\infty} \left(t_1^2 - \eta^2\right)^q \exp\left[-\beta\left(t_1^2 - \eta^2\right)\right] dt_1 \tag{C.6}$$

for $q = 0, 1, 2, \cdots, \infty$. Appendix D shows that (2.35) can be derived from (C.6).

Next consider the case for $N_{\min} > 1$.

C.3 Separation of the $N_{\min}$-fold Integration for $N_{\min} > 1$

It can be shown that an integration of an integrand which involves $\prod_{1 \le i < j \le N_{\min}-1} (z_i - z_j)^2$ can be converted to an integration of an integrand containing the determinant of a matrix [50, Chapter 3]. By exploring this result, from (C.4) it follows that

$$\begin{aligned} & \sum_{m=1}^{N_{\min}} \int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\ = & \frac{K_1}{2^{N_{\min}}} z_0^{N-N_{\min}} \int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty} \left\{ \prod_{n=1}^{N_{\min}-1} (z_0 - z_n) \exp(-\beta z_n)\, z_n^{N_{\max}-N_{\min}} \right\} \det \mathbf{Z} \\ & \times \left\{ \int_{\eta}^{\infty} (1 - t_{N_{\min}}) \left(t_{N_{\min}}^2 - \eta^2\right)^{N_{\max}-N} \right. \\ & \times \left. \left[\prod_{n=1}^{N_{\min}-1} \left(t_{N_{\min}}^2 - \eta^2 - z_n\right) \right] \exp\left[-\beta\left(t_{N_{\min}}^2 - \eta^2\right)\right] dt_{N_{\min}} \right\} \\ & \times dz_{N_{\min}-1} \cdots dz_2 dz_1, \end{aligned} \tag{C.7}$$

where $\det \mathbf{Z}$ is the determinant of the matrix $\mathbf{Z}$, whose i-th row, j-th column element is z_j^{i+j-2}. Note that all the elements on the j-th column of the matrix $\mathbf{Z}$ depend only on variable z_j.

Obviously, $\prod_{n=1}^{2}(z_0 - z_n)$ can be expressed as

$$\prod_{n=1}^{2}(z_0 - z_n) = \sum_{p=0}^{2}(-1)^{2-p} z_0^p \sum_{\substack{m_1+m_2=2-p \\ m_1,m_2\in\{0,1\}}} z_1^{m_1} z_2^{m_2}. \tag{C.8}$$

The second summation is taken over all sets of m_1 and m_2 satisfying the stated conditions. Expanding the two products in (C.7) similarly,

$$\begin{aligned}
&\sum_{m=1}^{N_{\min}} \int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\
= \;& \frac{K_1}{2^{N_{\min}}} z_0^{N-N_{\min}} \sum_{p=0}^{N_{\min}-1} (-1)^{N_{\min}-1-p} z_0^p \sum_{q=0}^{N_{\min}-1} (-1)^{N_{\min}-1-q} \\
& \times \sum_{\substack{m_1+\cdots+m_{N_{\min}-1}=N_{\min}-1-p \\ m_i\in\{0,1\}}} \sum_{\substack{n_1+\cdots+n_{N_{\min}-1}=N_{\min}-1-q \\ n_i\in\{0,1\}}} \\
& \times \int_0^\infty \int_0^\infty \cdots \int_0^\infty \left\{ \prod_{j=1}^{N_{\min}-1} z_j^{N_{\max}-N_{\min}+m_j+n_j} \exp(-\beta z_j) \right\} \\
& \times \det \mathbf{Z}\; dz_{N_{\min}-1} \cdots dz_2 dz_1 \\
& \times \left\{ \int_\eta^\infty (1 - t_{N_{\min}}) \left(t_{N_{\min}}^2 - \eta^2\right)^{N_{\max}-N+q} \exp\left[-\beta\left(t_{N_{\min}}^2 - \eta^2\right)\right] dt_{N_{\min}} \right\}
\end{aligned} \tag{C.9}$$

Since the elements on the j-th column of matrix $\mathbf{Z}$ depend only on variable z_j, $z_j^{N_{\max}-N_{\min}+m_j+n_j} \exp(-\beta z_j)$ (for $j = 1, 2, \cdots, N_{\min} - 1$) can be multiplied with all elements of the j-th column of $\mathbf{Z}$. Then the j-th column is integrated with respect to z_j before the determinant is calculated. In this way, the $N_{\min}$-fold integration is separated into independent integrations. By carrying out these straightforward integrations and substituting K_1 from (2.33) and $z_0 = -1/\gamma$ in (C.9), after some manipulations, (2.42) is obtained.

When $N_{\min} = 1$, if assume $\det \mathbf{W} = 1$, then it can be shown that (2.42) yields the same result as (C.5). Therefore (2.42) can be used for $N_{\min} = 1$ as well as for $N_{\min} > 1$.

APPENDIX D

DERIVATION OF SERIES B_Q

In this appendix, the method of calculating the series B_q $(q = 0, 1, 2, \cdots)$ is derived.

In the following, the expressions for B_0 and B_1 are derived first. Then it will be shown that for $q \geq 2$, B_q could be evaluated by B_{q-1} and B_{q-2}.

D.1 Evaluation of B_0

From (C.6),

$$\begin{aligned} B_0 &= \int_{\eta}^{\infty} \exp\left(-\beta\left(t^2 - \eta^2\right)\right) dt \\ &= \exp\left(\beta\eta^2\right) \int_{\eta}^{\infty} \exp\left(-\beta t^2\right) dt, \end{aligned} \tag{D.1}$$

which can be easily shown to be equal to

$$B_0 = \sqrt{\frac{\pi}{\beta}} \exp\left(\beta\eta^2\right) Q\left(\sqrt{2\beta}\eta\right), \tag{D.2}$$

where $Q(\cdot)$ is the Gaussian Q-function.

D.2 Evaluation of B_1

Again from (C.6),

$$\begin{aligned} B_1 &= \int_{\eta}^{\infty} \left(t^2 - \eta^2\right) \exp\left(-\beta\left(t^2 - \eta^2\right)\right) dt \\ &= \exp\left(\beta\eta^2\right) \left[\int_{\eta}^{\infty} t^2 \exp\left(-\beta t^2\right) dt - \eta^2 \int_{\eta}^{\infty} \exp\left(-\beta t^2\right) dt\right]. \end{aligned} \tag{D.3}$$

Define $\frac{1}{2}z^2 = \beta t^2$, then $t = \frac{1}{\sqrt{2\beta}} z$, $z = \sqrt{2\beta}t$, and

$$\begin{aligned} B_1 &= \exp\left(\beta\eta^2\right) \left[\int_{\sqrt{2\beta}\eta}^{\infty} \frac{1}{2\beta} z^2 \exp\left(-\frac{1}{2}z^2\right) \frac{1}{\sqrt{2\beta}} dz - \eta^2 \int_{\sqrt{2\beta}\eta}^{\infty} \exp\left(-\frac{1}{2}z^2\right) \frac{1}{\sqrt{2\beta}} dz\right] \\ &= \frac{1}{\sqrt{2\beta}} \exp\left(\beta\eta^2\right) \left[\frac{1}{2\beta} \int_{\sqrt{2\beta}\eta}^{\infty} \exp\left(-\frac{1}{2}z^2\right) z^2 dz - \eta^2 \int_{\sqrt{2\beta}\eta}^{\infty} \exp\left(-\frac{1}{2}z^2\right) dz\right]. \end{aligned}$$

After some manipulations,

$$B_1 = \frac{\eta}{2\beta} + \sqrt{\frac{\pi}{\beta}}\left[\frac{1}{2\beta} - \eta^2\right]\exp\left(\beta\eta^2\right)Q\left(\sqrt{2\beta}\eta\right).$$

D.3 Evaluation of B_q for $q \geq 2$

For $q \geq 2$, the following proves that B_q can be expressed as a function of B_{q-1} and B_{q-2}:

$$\begin{aligned}
B_q &= \int_\eta^\infty \left(t^2-\eta^2\right)^{q-1}\left(t^2-\eta^2\right)\exp\left(-\beta\left(t^2-\eta^2\right)\right)dt \\
&= \int_\eta^\infty \left(t^2-\eta^2\right)^{q-1} t^2 \exp\left(-\beta\left(t^2-\eta^2\right)\right)dt \\
&\quad -\eta^2\int_\eta^\infty \left(t^2-\eta^2\right)^{q-1}\exp\left(-\beta\left(t^2-\eta^2\right)\right)dt \\
&= \int_\eta^\infty \left(t^2-\eta^2\right)^{q-1}\frac{1}{-2\beta}t d\exp\left(-\beta\left(t^2-\eta^2\right)\right) - \eta^2 B_{q-1} \\
&= \left(t^2-\eta^2\right)^{q-1}\frac{1}{-2\beta}t\exp\left(-\beta\left(t^2-\eta^2\right)\right)\Big|_\eta^\infty \\
&\quad +\frac{1}{2\beta}\int_\eta^\infty \left[(q-1)\left(t^2-\eta^2\right)^{q-2}2t^2 + \left(t^2-\eta^2\right)^{q-1}\right] \\
&\quad \exp\left(-\beta\left(t^2-\eta^2\right)\right)dt - \eta^2 B_{q-1}, \\
&= \frac{1}{2\beta}\int_\eta^\infty \left[2(q-1)\left(t^2-\eta^2\right)^{q-2}\left(t^2-\eta^2+\eta^2\right) + \left(t^2-\eta^2\right)^{q-1}\right] \\
&\quad \times\exp\left(-\beta\left(t^2-\eta^2\right)\right)dt - \eta^2 B_{q-1} \\
&= \frac{1}{2\beta}\int_\eta^\infty \left[(2q-1)\left(t^2-\eta^2\right)^{q-1} + 2(q-1)\eta^2\left(t^2-\eta^2\right)^{q-2}\right] \\
&\quad \times\exp\left(-\beta\left(t^2-\eta^2\right)\right)dt - \eta^2 B_{q-1} \\
&= \frac{1}{2\beta}\left[(2q-1)B_{q-1} + 2(q-1)\eta^2 B_{q-2}\right] - \eta^2 B_{q-1} \\
&= \left[\frac{(2q-1)}{2\beta} - \eta^2\right]B_{q-1} + \frac{(q-1)}{\beta}\eta^2 B_{q-2}. \qquad \text{(D.4)}
\end{aligned}$$

APPENDIX E

DERIVATION OF $B_K(Y_M)$ AS A SUMMATION OF $(Y_M^2 - \eta^2)$ TO INTEGER POWER

In this appendix, (2.43), which expresses the function $b_k(y_m)$ (defined in (2.30)) as a summation of $(y_m^2 - \eta^2)$ to integer power, is proven.

For notation simplicity, define

$$b_k = -(1+y_m)(\eta - y_m)^k + (1-y_m)(\eta + y_m)^k. \tag{E.1}$$

Then

$$b_0 = -2y_m \tag{E.2}$$

$$b_1 = 2(1-\eta)y_m. \tag{E.3}$$

By substituting in b_{k-1} and b_{k-2}, it could be easily proven that

$$b_k = 2\eta b_{k-1} + \left(y_m^2 - \eta^2\right) b_{k-2}. \tag{E.4}$$

To further simplify the notations, define mathematical symbols $P = 2\eta$, $Q = y_m^2 - \eta^2$. Then (E.4) becomes

$$b_k = Pb_{k-1} + Qb_{k-2}. \tag{E.5}$$

Next, try to find the relation between b_k and b_0, b_1. From (E.5),

$$\begin{aligned} b_k &= Pb_{k-1} + Qb_{k-2} \\ &= P(Pb_{k-2} + Qb_{k-3}) + Qb_{k-2} \\ &= \left(P^2 + Q\right) b_{k-2} + PQb_{k-3}. \end{aligned} \tag{E.6}$$

Continue on in this way,

$$b_k = \left(P^3 + 2PQ\right) b_{k-3} + \left(P^2Q + Q^2\right) b_{k-4} \tag{E.7}$$

$$b_k = \left(P^4 + 2P^2Q + P^2Q + Q^2\right) b_{k-4} + \left(P^3Q + 2PQ^2\right) b_{k-5}. \tag{E.8}$$

Judging from (E.5) to (E.8), it can be predicted that the relation between b_k and b_{k-I}, b_{k-I-1} (I is any positive integer) is:

- When I is odd,

$$b_k = \left[\sum_{t=0}^{O_I} \binom{I-t}{t} P^{I-2t}Q^t\right] b_{k-I} + \left[\sum_{t=0}^{O_I} \binom{I-1-t}{t} P^{I-2t-1}Q^{t+1}\right] b_{k-I-1}. \tag{E.9}$$

  where $O_I = (I-1)/2$.

- When I is even,

$$b_k = \left[\sum_{t=0}^{O_I} \binom{I-t}{t} P^{I-2t}Q^t\right] b_{k-I} + \left[\sum_{t=0}^{O_I-1} \binom{I-1-t}{t} P^{I-2t-1}Q^{t+1}\right] b_{k-I-1}. \tag{E.10}$$

  where $O_I = I/2$.

(E.5) to (E.8) have shown that (E.9) and (E.10) are valid for $I = 1, 2, 3, 4$. The validity of (E.9) and (E.10) for any positive integer I could be proven easily by the method of mathematical induction.

From (E.9) and (E.10), the relation between b_k and b_1, b_0 can be obtained as follows:

- When k is even,

  let $I = k - 1$ and substitute it into (E.9),

$$b_k = \left[\sum_{t=0}^{O_I} \binom{k-1-t}{t} P^{k-1-2t}Q^t\right] b_1 + \left[\sum_{t=0}^{O_I} \binom{k-1-1-t}{t} P^{k-1-2t-1}Q^{t+1}\right] b_0. \tag{E.11}$$

Define $T \triangleq k/2$. Since $I = 2O_I + 1$,

$$O_I = \frac{I-1}{2} = \frac{k-1-1}{2} = \frac{k}{2} - 1 = T - 1. \tag{E.12}$$

Substitute (E.12) into (E.11). The relation between b_k and b_1, b_0 can be obtained as

$$b_k = \left[\sum_{t=0}^{T-1} \binom{k-1-t}{t} P^{k-1-2t} Q^t\right] b_1 + \left[\sum_{t=0}^{T-1} \binom{k-1-1-t}{t} P^{k-1-2t-1} Q^{t+1}\right] b_0. \tag{E.13}$$

- When k is odd,

  similarly as above,

$$b_k = \left[\sum_{t=0}^{T_1} \binom{k-1-t}{t} P^{k-1-2t} Q^t\right] b_1 + \left[\sum_{t=0}^{T_1-1} \binom{k-2-t}{t} P^{k-2-2t} Q^{t+1}\right] b_0, \tag{E.14}$$

where $T_1 = (k-1)/2$.

Finally, by substituting $P = 2\eta$, $Q = y_m^2 - \eta^2$, $b_0 = -2y_m$, and $b_1 = 2(1-\eta) y_m$ in (E.13) and (E.14), and after some manipulations,

$$b_k(y_m) = 2y_m \sum_{t=0}^{[k/2]} a_{k,t} \left(y_m^2 - \eta^2\right)^t, \tag{E.15}$$

where $[k/2]$ denotes the largest integer that is less than $k/2$. And $a_{k,t}$ is calculated differently for when k is even or odd as follows:

- When k is even,

$$a_{k,t} = \begin{cases} (2\eta)^{k-1}(1-\eta) & t = 0 \\ \left[\binom{k-1-t}{t}(1-\eta) - 2\eta\binom{k-1-t}{t-1}\right](2\eta)^{k-1-2t} & 1 \le t \le [k/2]-1 \\ -1 & t = [k/2] \end{cases}. \tag{E.16}$$

- When k is odd,

$$a_{k,t} = \begin{cases} (2\eta)^{k-1}(1-\eta) & t = 0 \\ \left[\binom{k-1-t}{t}(1-\eta) - 2\eta\binom{k-1-t}{t-1}\right](2\eta)^{k-1-2t} & 1 \le t \le [k/2] \end{cases}. \tag{E.17}$$

Assume $\binom{m}{n} = 0$ for $m < n$ or $n < 0$, (E.16) and (E.17) can be expressed as a single expression:

$$a_{k,t} = \left[\binom{k-1-t}{t} (1-\eta) - 2\eta \binom{k-1-t}{t-1} \right] (2\eta)^{k-1-2t},$$

for $0 \leq t \leq [k/2]$.

APPENDIX F

EVALUATION OF Υ_Q

In this appendix, the integral defined in (3.20) is evaluated. Substitute the function $V(\cdot)$ from (3.18) in (3.20),

$$\Upsilon_q = \int_0^\infty D\left(z_{N_{\min}}\right) f_q\left(z_{N_{\min}}\right) dz_{N_{\min}}, \tag{F.1}$$

where

$$D\left(z_{N_{\min}}\right) = \frac{1}{\pi}\int_0^\phi \left(\frac{\sin^2\theta}{\sin^2\theta+\xi_1}\right)^{N_A-N_{\min}} \frac{\left(\sin^2\theta\right)^{N_{\min}}}{\sin^2\theta+\xi_2} d\theta \tag{F.2}$$

$$\xi_1 = \xi\gamma \tag{F.3}$$

$$\xi_2 = \xi\frac{P_s}{P_I z_{N_{\min}}+\sigma^2} \tag{F.4}$$

$$f_q\left(z_{N_{\min}}\right) = z_{N_{\min}}^{N_{\max}-N_{\min}+q}\left(z_{N_{\min}}+\frac{\sigma^2}{P_I}\right)^{N_{\min}-1} e^{-z_{N_{\min}}}. \tag{F.5}$$

First evaluate $D\left(z_{N_{\min}}\right)$, which involves the integration over variable θ. The purpose is to express $D\left(z_{N_{\min}}\right)$ without integration.

F.1 Evaluation of $D\left(z_{N_{\min}}\right)$

From (F.2),

$$D\left(z_{N_{\min}}\right) = \frac{1}{\pi}\int_0^\phi \frac{\left(\sin^2\theta+\xi_1-\xi_1\right)^{N_A}}{\left(\sin^2\theta+\xi_1\right)^{N_A-N_{\min}}}\frac{1}{\sin^2\theta+\xi_2} d\theta. \tag{F.6}$$

By using the binomial expansion,

$$D\left(z_{N_{\min}}\right) = \sum_{k=0}^{N_A}\binom{N_A}{k}(-\xi_1)^k \frac{1}{\pi}\int_0^\phi \frac{1}{\sin^2\theta+\xi_2}\left(\sin^2\theta+\xi_1\right)^{N_{\min}-k} d\theta. \tag{F.7}$$

Separate the summation into two parts according to whether $N_{\min}-k$ is non-negative or negative.

$$D\left(z_{N_{\min}}\right) = \sum_{k=0}^{N_{\min}} \binom{N_A}{k} \left(-\xi_1\right)^k E_{N_{\min}-k}\left(\xi_1,\xi_2\right) + \sum_{k=N_{\min}+1}^{N_A} \binom{N_A}{k} \left(-\xi_1\right)^k U_{k-N_{\min}}\left(\xi_1,\xi_2\right), \tag{F.8}$$

where

$$E_m\left(\xi_1,\xi_2\right) = \frac{1}{\pi}\int_0^{\phi} \frac{1}{\sin^2\theta+\xi_2}\left(\sin^2\theta+\xi_1\right)^m d\theta \tag{F.9}$$

$$U_m\left(\xi_1,\xi_2\right) = \frac{1}{\pi}\int_0^{\phi} \frac{1}{\sin^2\theta+\xi_2}\frac{1}{\left(\sin^2\theta+\xi_1\right)^m} d\theta. \tag{F.10}$$

F.1.1 Evaluation of $E_m\left(\xi_1,\xi_2\right)$

For $m=0$,

$$E_0\left(\xi_2\right) = \frac{1}{\pi}\int_0^{\phi} \frac{1}{\sin^2\theta+\xi_2} d\theta = \frac{1}{\pi}\frac{1}{\sqrt{\xi_2\left(\xi_2+1\right)}}\operatorname{arctg}\left(\sqrt{\frac{\xi_2+1}{\xi_2}}\operatorname{tg}\phi\right), \tag{F.11}$$

where the result from [31, Eq. 2.562] is used. For $m \geq 1$, it can be shown that

$$E_m\left(\xi_1,\xi_2\right) = F_{m-1}\left(\xi_1\right) + \left(\xi_1-\xi_2\right) E_{m-1}\left(\xi_1,\xi_2\right), \tag{F.12}$$

where

$$F_m\left(\xi_1\right) = \frac{1}{\pi}\int_0^{\phi} \left(\sin^2\theta+\xi_1\right)^m d\theta. \tag{F.13}$$

By using the binomial expansion and [31, Eq. 2.513.1], the expression for F_m can be obtained as shown in (3.22). When (F.12) is expanded further,

$$E_m\left(\xi_1,\xi_2\right) = \sum_{i=1}^{m} \left(\xi_1-\xi_2\right)^{i-1} F_{m-i}\left(\xi_1\right) + \left(\xi_1-\xi_2\right)^m E_0\left(\xi_2\right), \tag{F.14}$$

which shows $E_m(\xi_1, \xi_2)$ can be evaluated from F_{m-i} and E_0.

F.1.2 Evaluation of $U_m(\xi_1, \xi_2)$

Similarly to the evaluation of $E_m(\xi_1, \xi_2)$,

$$U_m(\xi_1, \xi_2) = -\sum_{i=0}^{m-1} \left(\frac{1}{\xi_1 - \xi_2}\right)^{i+1} G_{m-i}(\xi_1) + \left(\frac{1}{\xi_1 - \xi_2}\right)^m E_0(\xi_2), \tag{F.15}$$

where

$$G_m(\xi_1) = \frac{1}{\pi} \int_0^{\phi} \frac{1}{\left(\sin^2\theta + \xi_1\right)^m} d\theta. \tag{F.16}$$

By using Eq. (29) and (40) in [51], the expression for $G_m(\xi_1)$ is obtained as shown in (3.25).

F.1.3 Summary for $D(z_{N_{\min}})$

Substitute (F.14) and (F.15) in (F.8). The expression for $D(z_{N_{\min}})$ is obtained as

$$\begin{aligned}
& D(z_{N_{\min}}) \\
= & \sum_{k=0}^{N_{\min}} \binom{N_A}{k} (-\xi_1)^k \\
& \times \left[\sum_{i=1}^{N_{\min}-k} (\xi_1 - \xi_2)^{i-1} F_{N_{\min}-k-i}(\xi_1) + (\xi_1 - \xi_2)^{N_{\min}-k} E_0(\xi_2)\right] \\
& + \sum_{k=N_{\min}+1}^{N_A} \binom{N_A}{k} (-\xi_1)^k \left[-\sum_{i=0}^{k-N_{\min}-1} \left(\frac{1}{\xi_1 - \xi_2}\right)^{i+1} G_{k-N_{\min}-i}(\xi_1)\right. \\
& \left. + \left(\frac{1}{\xi_1 - \xi_2}\right)^{k-N_{\min}} E_0(\xi_2)\right],
\end{aligned} \tag{F.17}$$

which does not contain any integral forms.

F.2 Evaluation of Υ_q

Substitute (F.17) into (F.1). Then

$$
\begin{aligned}
&\Upsilon_q \\
&= \sum_{k=0}^{N_{\min}} \binom{N_A}{k} (-\xi_1)^k \\
&\left[\sum_{i=1}^{N_{\min}-k} F_{N_{\min}-k-i}(\xi_1) \int_0^{\infty} (\xi_1 - \xi_2)^{i-1} f_q(z_{N_{\min}})\, dz_{N_{\min}} \right. \\
&\left. + \int_0^{\infty} (\xi_1 - \xi_2)^{N_{\min}-k} E_0(\xi_2) f_q(z_{N_{\min}})\, dz_{N_{\min}} \right] + \sum_{k=N_{\min}+1}^{N_A} \binom{N_A}{k} \\
&\times (-\xi_1)^k \left[- \sum_{i=0}^{k-N_{\min}-1} G_{k-N_{\min}-i}(\xi_1) \int_0^{\infty} \left(\frac{1}{\xi_1 - \xi_2} \right)^{i+1} f_q(z_{N_{\min}})\, dz_{N_{\min}} \right. \\
&\left. + \int_0^{\infty} \left(\frac{1}{\xi_1 - \xi_2} \right)^{k-N_{\min}} E_0(\xi_2) f_q(z_{N_{\min}})\, dz_{N_{\min}} \right]. \qquad \text{(F.18)}
\end{aligned}
$$

Substituting $f_q(z_{N_{\min}})$ (from (F.5)), E_0 (from (F.11)) and ξ_2 (from (F.4)) into (F.18), after some straightforward manipulations, Υ_q is obtained as shown in (3.21).

APPENDIX G

INTEGRATION OF I_0

In this appendix, the following integration of zeroth order modified Bessel function is proven:

$$\int_0^\infty te^{-\alpha t^2} I_0(2\sqrt{\beta}t)dt = \frac{1}{2\alpha}e^{\frac{\beta}{\alpha}}. \tag{G.1}$$

Eq. (6.614.3) in [31] is

$$\int_0^\infty e^{-\alpha x} I_{2\nu}(2\sqrt{\beta x})dx = \frac{e^{\frac{1}{2}\frac{\beta}{\alpha}}}{\sqrt{\alpha\beta}}\frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)}M_{-\frac{1}{2},\nu}(\frac{\beta}{\alpha}). \tag{G.2}$$

Let $\nu = 0$,

$$\int_0^\infty e^{-\alpha x} I_0(2\sqrt{\beta x})dx = \frac{e^{\frac{1}{2}\frac{\beta}{\alpha}}}{\sqrt{\alpha\beta}}M_{-\frac{1}{2},0}(\frac{\beta}{\alpha}). \tag{G.3}$$

By using equation of $M_{-\mu-\frac{1}{2},\mu}(x)$ in [52, P. 432],

$$\int_0^\infty e^{-\alpha x} I_0(2\sqrt{\beta x})dx = \frac{e^{\frac{1}{2}\frac{\beta}{\alpha}}}{\sqrt{\alpha\beta}}(\frac{\beta}{\alpha})^{\frac{1}{2}}e^{\frac{1}{2}\frac{\beta}{\alpha}} = \frac{1}{\alpha}e^{\frac{\beta}{\alpha}}. \tag{G.4}$$

Define $x = t^2$,

$$\int_0^\infty e^{-\alpha x} I_0(2\sqrt{\beta x})dx = \int_0^\infty 2te^{-\alpha t^2} I_0(2\sqrt{\beta}t)dt = \frac{1}{\alpha}e^{\frac{\beta}{\alpha}}. \tag{G.5}$$

It follows that

$$\int_0^\infty te^{-\alpha t^2} I_0(2\sqrt{\beta}t)dt = \frac{1}{2\alpha}e^{\frac{\beta}{\alpha}}. \tag{G.6}$$

APPENDIX H

DERIVATION OF THE CHARACTERISTIC FUNCTION FOR MSDD

In this appendix, the expression (5.31) for the characteristic function $\Phi_{D|\boldsymbol{\lambda}}(j\omega)$ of the MSDD test statistic D is derived. To that end, by using (5.23) and (5.29), the test statistic D can be expressed in the following quadratic form:

$$D = \sum_{m=1}^{N_A} b_m^2 \left(|y_m(\mathbf{s}_k)|^2 - |y_m(\mathbf{s}'_k)|^2 \right), \tag{H.1}$$

where

$$b_m = \sqrt{\frac{1}{\lambda_m \left(K P_s + \lambda_m \right)}} \tag{H.2}$$

for $m = 1, 2, \cdots, N_A$. Define

$$d_m = b_m^2 \left(|y_m(\mathbf{s}_k)|^2 - |y_m(\mathbf{s}'_k)|^2 \right), \tag{H.3}$$

then

$$D = \sum_{m=1}^{N_A} d_m. \tag{H.4}$$

Also define vector $\mathbf{y}(\mathbf{s}_k) = [y_1(\mathbf{s}_k), y_2(\mathbf{s}_k), \cdots, y_{N_A}(\mathbf{s}_k)]^T$, then from (5.12) and (5.8),

$$\mathbf{y}(\mathbf{s}_k) = \left(\sum_{i=0}^{K-1} \mathbf{x}_{k-i} s_{k-i}^* \right) = \mathbf{U}_z^H \left(\sum_{i=0}^{K-1} \mathbf{r}_{k-i} s_{k-i}^* \right). \tag{H.5}$$

From the signal model in Section II, $E[\mathbf{y}(\mathbf{s}_k)] = E[\mathbf{y}(\mathbf{s}'_k)] = 0$. After some algebra, the covariance matrix of $\mathbf{y}(\mathbf{s}_k)$ can be evaluated as

$$\begin{aligned} E\left[\mathbf{y}(\mathbf{s}_k) \mathbf{y}(\mathbf{s}_k)^H \right] &= E\left[\mathbf{U}^H \left(\sum_{i=0}^{K-1} \mathbf{r}_{k-i} s_{k-i}^* \right) \left(\sum_{m=0}^{K-1} \mathbf{r}_{k-m} s_{k-m}^* \right)^H \mathbf{U} \right] \\ &= K^2 P_s \mathbf{I}_{N_A} + K\boldsymbol{\Lambda}. \end{aligned} \tag{H.6}$$

Similarly,

$$E\left[\mathbf{y}\left(\mathbf{s}_k'\right)\mathbf{y}\left(\mathbf{s}_k'\right)^H\right] = \left|v(\mathbf{s}_k,\mathbf{s}_k')\right|^2 P_s\mathbf{I}_{N_A} + K\mathbf{\Lambda} \tag{H.7}$$

$$E\left[\mathbf{y}\left(\mathbf{s}_k\right)\mathbf{y}\left(\mathbf{s}_k'\right)^H\right] = v^*(\mathbf{s}_k,\mathbf{s}_k')\left(KP_s\mathbf{I}_{N_A} + \mathbf{\Lambda}\right) \tag{H.8}$$

$$E\left[\mathbf{y}\left(\mathbf{s}_k'\right)\mathbf{y}\left(\mathbf{s}_k\right)^H\right] = v(\mathbf{s}_k,\mathbf{s}_k')\left(KP_s\mathbf{I}_{N_A} + \mathbf{\Lambda}\right), \tag{H.9}$$

where $v(\mathbf{s}_k,\mathbf{s}_k') = \mathbf{s}_k'^H\mathbf{s}_k$.

To use the results in [2, Appendix B], identify the following quantities using the notation in the reference: $X_m = y_m(\mathbf{s}_k)$, $Y_m = y_m(\mathbf{s}_k')$. Then from (H.6) to (H.9), in the notation of [2, Appendix B],

$$\overline{X}_m = \overline{Y}_m = 0 \tag{H.10}$$

$$\mu_{xx,m} = \frac{1}{2}\left(K^2P_s + K\lambda_m\right) \tag{H.11}$$

$$\mu_{yy,m} = \frac{1}{2}\left(\left|v(\mathbf{s}_k,\mathbf{s}_k')\right|^2 P_s + K\lambda_m\right) \tag{H.12}$$

$$\mu_{xy,m} = \frac{1}{2}v^*(\mathbf{s}_k,\mathbf{s}_k')\left(KP_s + \lambda_m\right) \tag{H.13}$$

$$\mu_{yx,m} = \frac{1}{2}v(\mathbf{s}_k,\mathbf{s}_k')\left(KP_s + \lambda_m\right). \tag{H.14}$$

Substitute the above equations into Eq. (B-6) and Eq. (B-5) in [2, Appendix B], together with $A = b_m^2$, $B = -A_m^2$, and $C_m = 0$. After some straightforward manipulations, the characteristic function of d_m is obtained as

$$\phi_{d_m|\boldsymbol{\lambda}}(j\omega) = \frac{\omega_{1,m}\,\omega_{2,m}}{\left(\omega - \omega_{1,m}\right)\left(\omega - \omega_{2,m}\right)}, \tag{H.15}$$

where

$$\omega_{1,m} = -j\frac{1}{2\zeta}\left[\sqrt{\zeta^2P_s^2 + 4\left(KP_s + \lambda_m\right)\lambda_m\zeta} - \zeta P_s\right] \tag{H.16}$$

$$\omega_{2,m} = j\frac{1}{2\zeta}\left[\sqrt{\zeta^2P_s^2 + 4\left(KP_s + \lambda_m\right)\lambda_m\zeta} + \zeta P_s\right], \tag{H.17}$$

and

$$\zeta = K^2 - \left|v(\mathbf{s}_k,\mathbf{s}_k')\right|^2. \tag{H.18}$$

It follows that the characteristic function of D is

$$\Phi_{D|\boldsymbol{\lambda}}(j\omega) = \prod_{m=1}^{N_A} \phi_{d_m|\boldsymbol{\lambda}}(j\omega) = \prod_{m=1}^{N_A} \frac{\omega_{1,m}\,\omega_{2,m}}{(\omega - \omega_{1,m})(\omega - \omega_{2,m})}. \tag{H.19}$$

Remember that for a system with N_I interference sources, the eigenvalues of the interference plus noise covariance matrix are $\lambda_m = \sigma^2$ for $m = N_{\min}+1$, $N_{\min}+2$, $\cdots$, N_A. Define

$$\omega_1 \triangleq \omega_{1,N_{\min}+1} = -j\frac{1}{2\zeta}\left[\sqrt{\zeta^2 P_s^2 + 4(KP_s + \sigma^2)\sigma^2\zeta} - \zeta P_s\right] \tag{H.20}$$

$$\omega_2 \triangleq \omega_{2,N_{\min}+1} = j\frac{1}{2\zeta}\left[\sqrt{\zeta^2 P_s^2 + 4(KP_s + \sigma^2)\sigma^2\zeta} + \zeta P_s\right]. \tag{H.21}$$

It follows that $\omega_{1,m} = \omega_1$, and $\omega_{2,m} = \omega_2$ for $m = N_{\min}+1, N_{\min}+2, \cdots, N_A$. Hence the characteristic function can be expressed as (5.31).

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